

# HEAT KERNEL ON SMOOTH METRIC MEASURE SPACES AND APPLICATIONS

JIA-YONG WU AND PENG WU

**ABSTRACT.** We derive a Harnack inequality for positive solutions of the  $f$ -heat equation and Gaussian upper and lower bounds for the  $f$ -heat kernel on complete smooth metric measure spaces  $(M, g, e^{-f} dv)$  with Bakry-Émery Ricci curvature bounded below. The lower bound is sharp. The main argument is the De Giorgi-Nash-Moser theory. As applications, we prove an  $L_f^1$ -Liouville theorem for  $f$ -subharmonic functions and an  $L_f^1$ -uniqueness theorem for  $f$ -heat equations when  $f$  has at most linear growth. We also obtain eigenvalues estimates and  $f$ -Green's function estimates for the  $f$ -Laplace operator.

## 1. INTRODUCTION

Heat kernel estimate is one of the fundamental problems in Riemannian geometry. For Riemannian manifolds, there are two classical methods for the heat kernel estimate. One is the gradient estimate technique developed by Li and Yau [26], using which they derived two-sided Gaussian bounds for the heat kernel on Riemannian manifolds with Ricci curvature bounded below. The other is the Moser iteration technique invented by Moser [31]. Grigor'yan [17] and Saloff-Coste [40, 41, 42] developed this technique and independently derived heat kernel estimates on Riemannian manifolds satisfying volume doubling property and the Poincaré inequality. There has been lots of work on improving heat kernel estimates on Riemannian manifolds, and generalizing heat kernel estimates to general spaces, see excellent surveys [18, 19, 42] and references therein.

In this paper we will investigate heat kernel estimates on smooth metric measure spaces and various applications. Let  $(M, g)$  be an  $n$ -dimensional complete Riemannian manifold, and let  $f$  be a smooth function on  $M$ . Then the triple  $(M, g, e^{-f} dv)$  is called a complete smooth metric measure space, where  $dv$  is the volume element of  $g$ , and  $e^{-f} dv$  (for short,  $d\mu$ ) is called the weighted volume element or the weighted measure. On a smooth metric measure space, the  $m$ -Bakry-Émery Ricci curvature [2, 39, 28] is defined by

$$\text{Ric}_f^m := \text{Ric} + \nabla^2 f - \frac{1}{m} df \otimes df,$$

where  $\text{Ric}$  is the Ricci curvature of  $(M, g)$ ,  $\nabla^2$  is the Hessian with respect to  $g$ , and  $m \in \mathbb{R} \cup \{\pm\infty\}$  (when  $m = 0$  we require  $f$  to be a constant).  $m$ -Bakry-Émery Ricci curvature is a natural generalization of Ricci curvature on Riemannian manifolds,

---

*Date:* June 24, 2014.

*2010 Mathematics Subject Classification.* Primary 35K08; Secondary 53C21, 58J35.

*Key words and phrases.* smooth metric measure space, Bakry-Émery Ricci curvature, heat kernel, Harnack inequality, Liouville theorem, eigenvalue, Green's function, parabolicity.

see [2, 3, 28, 29, 45] and references therein. In particular, a smooth metric measure space satisfying

$$\text{Ric}_f^m = \lambda g,$$

for some  $\lambda \in \mathbb{R}$ , is called an  $m$ -quasi-Einstein manifold (see [8]), which can be considered as natural generalization of Einstein manifold. When  $0 < m < \infty$ ,  $(M^n \times F^m, g_M + e^{-2\frac{f}{m}} g_F)$ , with  $(F^m, g_F)$  an Einstein manifold, is a warped product Einstein manifold. When  $m = 2 - n$ ,  $(M^n, g)$  is a conformally Einstein manifold, in fact  $\bar{g} = e^{\frac{f}{n-2}} g$  is the Einstein metric. When  $m = 1$ ,  $(M^n, g)$  is the so-called static manifold in general relativity. When  $m = \infty$ , we write

$$\text{Ric}_f = \text{Ric}_f^\infty,$$

and the quasi-Einstein equation reduces to a gradient Ricci soliton. The gradient Ricci soliton is called shrinking, steady, or expanding, if  $\lambda > 0$ ,  $\lambda = 0$ , or  $\lambda < 0$ , respectively. Ricci solitons play an important role in the Ricci flow and Perelman's resolution of Poincaré conjecture and geometrization conjecture, see [6, 21] and references therein for nice surveys.

On a smooth metric measure space  $(M, g, e^{-f} dv)$ , the  $f$ -Laplacian  $\Delta_f$  is defined as

$$\Delta_f = \Delta - \nabla f \cdot \nabla,$$

which is self-adjoint with respect to  $e^{-f} dv$ . The  $f$ -heat equation is defined as

$$(\partial_t - \Delta_f)u = 0.$$

We denote the  $f$ -heat kernel by  $H(x, y, t)$ , that is, for each  $y \in M$ ,  $H(x, y, t) = u(x, t)$  is the minimal positive solution of the  $f$ -heat equation satisfying the initial condition  $\lim_{t \rightarrow 0} u(x, t) = \delta_{f,y}(x)$ , where  $\delta_{f,y}(x)$  is the  $f$ -delta function defined by

$$\int_M \phi(x) \delta_{f,y}(x) e^{-f} dv = \phi(y)$$

for any  $\phi \in C_0^\infty(M)$ . Similarly a function  $u$  is said to be  $f$ -harmonic if  $\Delta_f u = 0$ , and  $f$ -subharmonic ( $f$ -superharmonic) if  $\Delta_f u \geq 0$  ( $\Delta_f u \leq 0$ ). It is easy to see that the absolute value of an  $f$ -harmonic function is a nonnegative  $f$ -subharmonic function. The weighted  $L^p$ -norm (or  $L_f^p$ -norm) is defined as

$$\|u\|_p = \left( \int_M |u|^p e^{-f} dv \right)^{1/p}$$

for any  $0 < p < \infty$ . We say that  $u$  is  $L_f^p$ -integrable, i.e.  $u \in L_f^p$ , if  $\|u\|_p < \infty$ .

Recall that for Riemannian manifolds, using the classical Bochner formula, Li-Yau [26] derived the gradient estimate and heat kernel estimate. For smooth metric measure spaces with  $m < \infty$ , there is an analogue of Bochner formula for  $\text{Ric}_f^m$ ,

$$\begin{aligned} (1.1) \quad \frac{1}{2} \Delta_f |\nabla u|^2 &= |\nabla^2 u|^2 + \langle \nabla \Delta_f u, \nabla u \rangle + \text{Ric}_f^m(\nabla u, \nabla u) + \frac{1}{m} |\langle \nabla f, \nabla u \rangle|^2 \\ &\geq \frac{(\Delta_f u)^2}{m+n} + \langle \nabla \Delta_f u, \nabla u \rangle + \text{Ric}_f^m(\nabla u, \nabla u). \end{aligned}$$

Therefore when  $m < \infty$ , the Bochner formula for  $\text{Ric}_f^m$  can be considered as the Bochner formula for the Ricci tensor of an  $(n+m)$ -dimensional manifold, and for smooth metric measure spaces with  $\text{Ric}_f^m$  bounded below, one has nice  $f$ -mean curvature comparison and  $f$ -volume comparison theorems which are similar to classical

ones for Riemannian manifolds, see [3, 45], in particular, the comparison theorems do not depend on  $f$ ; X.-D. Li [27] derived an analogue of Li-Yau gradient estimate, using which he proved  $f$ -heat kernel estimates and several Liouville theorems; and in [9], by analyzing a family of warped product manifolds, Charalambous and Z. Lu obtained  $f$ -heat kernel estimates and essential spectrum.

Unfortunately when  $m = \infty$ , due to the lack of the extra term  $\frac{1}{m}|\langle \nabla f, \nabla u \rangle|^2$  in the Bochner formula (1.1), one can only derive local  $f$ -mean curvature comparison and local  $f$ -volume comparison (see [45]) which highly rely on the potential function  $f$ , and this makes it much more difficult to investigate smooth metric measure spaces with  $\text{Ric}_f$  bounded below. According to [34, 35], there seems essential obstacles to derive Li-Yau gradient estimate directly using the Bochner formula (1.1), even with strong growth assumption on  $f$ . It is interesting to point out that for  $f$ -harmonic functions, Munteanu and Wang [34, 35] obtained Yau's gradient estimate using both Yau's idea and the De Giorgi-Nash-Moser theory, under appropriate assumptions on  $f$ .

In this paper we observe that, without any assumption on  $f$ , one can derive a Harnack inequality for positive solutions of  $f$ -heat equation, and local Gaussian bounds for the  $f$ -heat kernel on smooth metric measure spaces using the De Giorgi-Nash-Moser theory. Moreover, similar to [34, 35], in each step one needs to figure out the accurate coefficients, which play key roles in the applications. As applications, we prove a Liouville theorem for  $f$ -subharmonic functions, eigenvalues estimates for the  $f$ -Laplacian, and  $f$ -Green's functions estimates.

Let us first state the local  $f$ -heat kernel estimates,

**Theorem 1.1.** *Let  $(M, g, e^{-f} dv)$  be an  $n$ -dimensional complete noncompact smooth metric measure space with  $\text{Ric}_f \geq -(n-1)K$  for some constant  $K \geq 0$ . For any point  $o \in M$  and  $R > 0$ , denote*

$$A(R) = \sup_{x \in B_o(3R)} |f(x)|, \quad A'(R) = \sup_{x \in B_o(3R)} |\nabla f(x)|.$$

*Then for any  $\epsilon > 0$ , there exist constants  $c_1(n, \epsilon)$ ,  $c_i(n)$ ,  $2 \leq i \leq 6$  such that*

(1.2)

$$\begin{aligned} & \frac{c_1 e^{c_2 A + c_3(1+A)\sqrt{K}t}}{V_f(B_x(\sqrt{t})^{1/2} V_f(B_y(\sqrt{t})^{1/2})} \exp\left(-\frac{d^2(x, y)}{(4+\epsilon)t}\right) \\ & \geq H(x, y, t) \geq \frac{c_4 e^{-c_5(A'^2+K)t}}{V_f(B_x(\sqrt{t}))} \exp\left(-\frac{d^2(x, y)}{c_6 t}\right) \end{aligned}$$

*for all  $x, y \in B_o(\frac{1}{2}R)$  and  $0 < t < R^2/4$ .  $\lim_{\epsilon \rightarrow 0} c_1(n, \epsilon) = \infty$ .*

When  $f$  is bounded, the first author [47] obtained  $f$ -heat kernel upper and lower bounds estimates. When  $\text{Ric}_f \geq 0$ , the authors [48] obtained  $f$ -heat kernel upper bound estimates without assumptions on  $f$ .

It is worthy to point out that the lower bound estimate is sharp. Indeed, let  $(\mathbb{R}, g_0, e^{-f} dx)$  be a 1-dimensional steady Gaussian soliton, where  $g_0$  is the Euclidean metric and  $f(x) = \pm x$ . From [48] the  $f$ -heat kernel is given by

$$H(x, y, t) = \frac{e^{\pm \frac{x+y}{2}} \cdot e^{-t/4}}{(4\pi t)^{1/2}} \times \exp\left(-\frac{|x-y|^2}{4t}\right).$$

Obviously, the lower bound estimate is achieved by the above  $f$ -heat kernel for steady Gaussian soliton as long as  $t$  is very large.

**Remark 1.2.** The factor  $A'$  in the lower bound estimate comes from Harnack inequality in Theorem 1.3, it will be more interesting to derive a new sharp lower bound in terms of  $A$  instead of  $A'$ , if possible.

The proof of upper bound estimate of the  $f$ -heat kernel uses a weighted mean value inequality and Davies's integral estimate [14]. The proof of lower bound estimate follows from a Harnack inequality and a chaining argument, while the proof of the Harnack inequality, follows from the arguments in [41, 42].

To state the Harnack inequality, let us first introduce notations, for any point  $x \in M$  and  $r > 0$ ,  $s \in \mathbb{R}$ , and  $0 < \varepsilon < \eta < \delta < 1$ , we denote  $B = B_x(r)$ ,  $\delta B = B_x(\delta r)$  and

$$Q = B \times (s - r^2, s), \quad Q_- = \delta B \times (s - \delta r^2, s - \eta r^2), \quad Q_+ = \delta B \times (s - \varepsilon r^2, s).$$

**Theorem 1.3.** *Let  $(M, g, e^{-f} dv)$  be an  $n$ -dimensional complete noncompact smooth metric measure space with  $\text{Ric}_f \geq -(n-1)K$  for some constant  $K \geq 0$ . Let  $u$  be a positive solution to the  $f$ -heat equation in  $Q$ , there exist constants  $c_1$  and  $c_2$  depending on  $n, \varepsilon, \eta$  and  $\delta$ , such that*

$$\sup_{Q_-} u \leq c_1 e^{c_2(A'^2+K)r^2} \inf_{Q_+} u,$$

where  $A'(r) = \sup_{y \in B_x(3r)} |\nabla f(y)|$ .

By a different volume comparison, we get another form of Harnack inequality and lower bound estimate for the  $f$ -heat kernel,

**Theorem 1.4.** *Under the assumptions of Theorem 1.3 and Theorem 1.1, respectively, we have*

$$\sup_{Q_-} \{u\} \leq \exp \left\{ c_1 e^{c_2 A} [(1 + A^2)Kr^2 + 1] \right\} \cdot \inf_{Q_+} \{u\},$$

where  $A = A(r) = \sup_{y \in B_x(3r)} |f(y)|$ , and

$$(1.3) \quad H(x, y, t) \geq \frac{c_4}{V_f(B_x(\sqrt{t}))} \times \exp \left[ -c_5 e^{c_6 A} \left( (1 + A^2)Kt + 1 + \frac{d^2(x, y)}{t} \right) \right],$$

for all  $x, y \in B_o(\frac{1}{2}R)$  and  $0 < t < R^2/4$ , where  $A = A(R) = \sup_{x \in B_o(3R)} |f(x)|$ . In particular, when  $f$  is bounded, we get

$$(1.4) \quad H(x, y, t) \geq \frac{c_1 e^{-c_2 Kt}}{V_f(B_x(\sqrt{t}))} \times \exp \left( -\frac{d^2(x, y)}{c_3 t} \right).$$

Next we derive several applications of the  $f$ -heat kernel estimates. First we prove a Liouville theorem for  $f$ -subharmonic functions. Recall Pigola, Rimoldi, Setti [38] proved that any nonnegative  $L_f^1$ -integrable  $f$ -superharmonic function must be constant if  $\text{Ric}_f$  is bounded below, without any assumption on  $f$ . However, as proved in [48], for  $f$ -subharmonic functions, the condition on  $f$  is necessary. In fact we provided explicit counterexamples illustrating that  $f$  cannot grow faster than quadratically when  $\text{Ric}_f \geq 0$ . Below we show that the  $L_f^1$ -Liouville theorem also holds for  $f$ -subharmonic functions when  $\text{Ric}_f \geq -(n-1)K$  and  $f$  has at most linear growth.

**Theorem 1.5.** *Let  $(M, g, e^{-f} dv)$  be an  $n$ -dimensional complete noncompact smooth metric measure space with  $\text{Ric}_f \geq -(n-1)K$  for some constant  $K > 0$ . Assume there exist nonnegative constants  $a$  and  $b$  such that*

$$|f|(x) \leq ar(x) + b,$$

*where  $r(x)$  is the distance function to a fixed point  $o \in M$ . Then any nonnegative  $L_f^1$ -integrable  $f$ -subharmonic function must be identically constant. In particular, any  $L_f^1$ -integrable  $f$ -harmonic function must be identically constant.*

There have been various Liouville type theorems for  $f$ -subharmonic and  $f$ -harmonic functions on smooth metric measure spaces and gradient Ricci solitons under different conditions, see Brighton [4], Cao-Zhou [7], Munteanu-Sesum [33], Munteanu-Wang [34, 35], Petersen-Wylie [37], and Wei-Wylie [45] for details.

By a similar argument in [23] (see also [48]), we also prove an  $L_f^1$ -uniqueness theorem for solutions of  $f$ -heat equation, see Theorem 5.3 in Section 5.

Second we derive lower bound estimates for eigenvalues of the  $f$ -Laplace operator on compact smooth metric measure spaces, by adapting the classical argument of Li-Yau [26],

**Theorem 1.6.** *Let  $(M, g, e^{-f} dv)$  be an  $n$ -dimensional compact smooth metric measure space with  $\text{Ric}_f \geq -(n-1)K$  for some constant  $K \geq 0$ . Let  $0 = \lambda_0 < \lambda_1 \leq \lambda_2 \leq \dots$  be eigenvalues of the  $f$ -Laplacian  $\Delta_f$ . Then there exists a constant  $C$  depending only on  $n$  and  $A = \max_{x \in M} f(x)$ , such that*

$$\begin{aligned} \lambda_k &\geq \frac{C(k+1)^{2/n}}{d^2}, & K = 0, \\ \lambda_k &\geq \frac{C}{d^2} \left( \frac{k+1}{\exp(C\sqrt{K}d)} \right)^{\frac{2}{n+4A}}, & K > 0, \end{aligned}$$

*for all  $k \geq 1$ , where  $d$  is the diameter of  $M$ .*

The upper bound estimates were proved by Hassannezhad [22], and Colbois, Soufi, Savo [12], which depend on norms of the potential function and conformal class of the metric. For the first eigenvalue, there have been more interesting results. When  $M$  is compact and  $\text{Ric}_f \geq \frac{a}{2} > 0$ , Andrews, Ni [1], and Futaki, Li, Li [16] derived lower bound estimates for the first eigenvalue, which depend on the diameter of the manifolds. When  $M$  is complete noncompact, Munteanu, Wang [34, 35, 36], and Wu [46] obtained first eigenvalues estimates under appropriate assumptions on  $f$ . Cheng, Zhou [11] proved an interesting Obata type theorem.

At last we discuss  $f$ -Green's functions estimates. We first get upper and lower estimates for  $f$ -Green's functions when  $f$  is bounded, which is similar to the classical estimates of Li-Yau [26] for Riemannian manifolds. Recall the  $f$ -Green's function on  $(M, g, e^{-f} dv)$  is defined as

$$G(x, y) = \int_0^\infty H(x, y, t) dt$$

if the integral on the right hand side converges. It is easy to check that  $G$  is positive and satisfies

$$\Delta_f G = -\delta_{f,y}(x).$$

**Theorem 1.7.** *Let  $(M, g, e^{-f} dv)$  be an  $n$ -dimensional complete noncompact smooth metric measure space with  $\text{Ric}_f \geq 0$  and  $f$  bounded. If  $G(x, y)$  exists, then there exist constants  $c_1$  and  $c_2$  depending only on  $n$  and  $\sup f$ , such that*

$$(1.5) \quad c_1 \int_{r^2}^{\infty} V_f^{-1}(B_x(\sqrt{t})) dt \leq G(x, y) \leq c_2 \int_{r^2}^{\infty} V_f^{-1}(B_x(\sqrt{t})) dt,$$

where  $r = r(x, y)$ .

Recently Dai, Sung, Wang, and Wei [13] observed that every gradient steady Ricci soliton admits a positive  $f$ -Green's function, hence it is  $f$ -nonparabolic. We provide an alternative proof using a criterion of Li-Tam [24, 25], and the  $f$ -heat kernel for steady Gaussian Ricci soliton,

**Theorem 1.8.** *Let  $(M^n, g, f)$  be a complete gradient steady soliton. Then there exists a positive smooth  $f$ -Green function, therefore the gradient steady soliton is  $f$ -nonparabolic.*

In [43], Song, Wei and Wu investigated several properties of  $f$ -Green's functions on smooth metric measure spaces. Pigola, Rimoldi, and Setti [38] proved that gradient shrinking Ricci solitons are  $f$ -parabolic.

The paper is organized as follows. In Section 2, we recall comparison theorems for the Bakry-Émery Ricci curvature bounded below, using which we derive a local  $f$ -volume doubling property, a local  $f$ -Neumann Poincaré inequality, a local Sobolev inequality and mean value inequalities for the  $f$ -heat equation. In Section 3, we prove a Moser's Harnack inequality of  $f$ -heat equation following the arguments of Saloff-Coste in [41, 42]. In Section 4, we prove local Gaussian upper and lower bound estimates of the  $f$ -heat kernel. In Section 5, following the same arguments of [48], we establish a new  $L_f^1$ -Liouville theorem for the  $f$ -harmonic function and a new  $L_f^1$ -uniqueness property for nonnegative solutions of the  $f$ -heat equation. In Section 6, we apply upper bounds of the  $f$ -heat kernel to get the eigenvalue estimates of the  $f$ -Laplacian on compact smooth metric measure spaces. In Section 7, we derive Green function estimates for smooth metric measure spaces with  $\text{Ric}_f \geq 0$  and  $f$  bounded, and for gradient steady Ricci solitons.

**Acknowledgement.** The authors thank Professors Xiaodong Cao and Laurent Saloff-Coste for their suggestions and great help. The second author thanks Professors Xianzhe Dai and Guofang Wei for helpful discussions, guidance, constant encouragement and support. The first author is partially supported by NSFC (11101267, 11271132) and the China Scholarship Council (201208310431). The second author is partially supported by an AMS-Simons travel grant.

## 2. POINCARÉ, SOBOLEV AND MEAN VALUE INEQUALITIES

Recall that for any point  $p \in M$  and  $R > 0$ , we denote

$$A(R) = A(p, R) = \sup_{x \in B_p(3R)} |f(x)|, \quad A'(R) = A'(p, R) = \sup_{x \in B_p(3R)} |\nabla f(x)|.$$

When there is no confusion we write  $A, A'$  for short. We start from the relative  $f$ -volume comparison theorem of Wei and Wylie [45].

**Lemma 2.1.** *Let  $(M, g, e^{-f} dv)$  be an  $n$ -dimensional complete noncompact smooth metric measure space. If  $\text{Ric}_f \geq -(n-1)K$  for some constant  $K \geq 0$ , then*

$$(2.1) \quad \frac{V_f(B_x(R_1, R_2))}{V_f(B_x(r_1, r_2))} \leq \frac{V_K^{n+4A}(B_x(R_1, R_2))}{V_K^{n+4A}(B_x(r_1, r_2))}$$

for any  $0 < r_1 < r_2$ ,  $0 < R_1 < R_2$ ,  $r_1 \leq R_1$ ,  $r_2 \leq R_2$ , where  $B_x(R_1, R_2) = B_x(R_2) \setminus B_x(R_1)$ , and  $A = A(x, \frac{1}{3}R_2)$ . Here  $V_K^{n+4A}(B_x(r))$  denotes the volume of the ball in the model space  $M_K^{n+4A}$ , i.e., the simply connected space form with constant sectional curvature  $-K$  and dimension  $n+4A$ .

Similarly we have

$$(2.2) \quad \frac{V_f(B_x(R_1, R_2))}{V_f(B_x(r_1, r_2))} \leq \frac{V_K^{n+4A'R_2}(B_x(R_1, R_2))}{V_K^{n+4A'R_2}(B_x(r_1, r_2))},$$

where  $A' = A'(x, \frac{1}{3}R_2)$ .

**Remark 2.2.** Following the proofs,  $A(R)$  in all following lemmas, propositions, theorems and corollaries can be replaced by  $RA'(R)$ . We will apply the first volume comparison (2.1) to derive heat kernel upper bound, and the second volume comparison (2.2) to derive Harnack inequality and heat kernel lower bound.

*Proof of Lemma 2.1.* Applying the weighted Bochner formula (1.1) and an ODE argument, Wei and Wylie (see (3.19) in [45]) proved the following  $f$ -mean curvature comparison theorem. Recall that the weighted mean curvature  $m_f(r)$  is defined as

$$m_f(r) = m(r) - \nabla f \cdot \nabla r = \Delta_f r.$$

If  $\text{Ric}_f \geq -(n-1)K$ , then

$$(2.3) \quad \begin{aligned} m_f(r) &\leq (n-1)\sqrt{K} \coth(\sqrt{K}r) + \frac{2K}{\sinh^2(\sqrt{K}r)} \int_0^r (f(t) - f(r)) \cosh(2\sqrt{K}t) dt \\ &\leq (n-1+4A)\sqrt{K} \cdot \coth(\sqrt{K}r) \end{aligned}$$

along any minimal geodesic segment from  $x$ . In geodesic polar coordinates, the volume element is written as

$$dv = \mathcal{A}(r, \theta) dr \wedge d\theta_{n-1},$$

where  $d\theta_{n-1}$  is the standard volume element of the unit sphere  $S^{n-1}$ . Let

$$\mathcal{A}_f(r, \theta) = e^{-f} \mathcal{A}(r, \theta).$$

By the first variation of the area,

$$\frac{\mathcal{A}'}{\mathcal{A}}(r, \theta) = (\ln(\mathcal{A}(r, \theta)))' = m(r, \theta).$$

Therefore

$$\frac{\mathcal{A}'_f}{\mathcal{A}_f}(r, \theta) = (\ln(\mathcal{A}_f(r, \theta)))' = m_f(r, \theta),$$

So for  $r < R$ ,

$$\frac{\mathcal{A}_f(R, \theta)}{\mathcal{A}_f(r, \theta)} \leq \frac{\mathcal{A}_K^{n+4A}(R)}{\mathcal{A}_K^{n+4A}(r)},$$

That is  $\frac{\mathcal{A}_f(r, \theta)}{\mathcal{A}_K^{n+4A}(r)}$  is nonincreasing in  $r$ , where  $\mathcal{A}_K^{n+4A}(r)$  is the volume element in the simply connected hyperbolic space of constant sectional curvature  $-K$  and dimension  $n + 4A$ . Applying Lemma 3.2 in [50], we get

$$\frac{\int_{R_1}^{R_2} \mathcal{A}_f(R, \theta) dt}{\int_{r_1}^{r_2} \mathcal{A}_f(r, \theta) dt} \leq \frac{\int_{R_1}^{R_2} \mathcal{A}_K^{n+4A}(R, \theta) dt}{\int_{r_1}^{r_2} \mathcal{A}_K^{n+4A}(r, \theta) dt}$$

for  $0 < r_1 < r_2$ ,  $0 < R_1 < R_2$ ,  $r_1 \leq R_1$  and  $r_2 \leq R_2$ . Integrating along the sphere direction proves (2.1).

The second volume comparison (2.2) follows from an observation for the weighted mean curvature,

$$(2.4) \quad \begin{aligned} m_f(r) &\leq (n-1)\sqrt{K} \coth(\sqrt{K}r) + \frac{2K}{\sinh^2(\sqrt{K}r)} \int_0^r (f(t) - f(r)) \cosh(2\sqrt{K}t) dt \\ &\leq (n-1+4A'r)\sqrt{K} \cdot \coth(\sqrt{K}r). \end{aligned}$$

□

Let  $V_K^{n+4A}(B_x(r))$  be the volume of the ball of radius  $r$  in the simply connected hyperbolic space of constant sectional curvature  $-K$  and dimension  $n + 4A$ . If  $K > 0$ , the model space is the hyperbolic space. If  $K = 0$ , the model space is the Euclidean space. In any case, we have the estimate

$$(2.5) \quad \omega_{n+4A} \cdot r^{n+4A} \leq V_K(B_x(r)) \leq \omega_{n+4A} \cdot r^{n+4A} e^{(n-1+4A)\sqrt{K}r}$$

where  $\omega_{n+4A}$  is the volume of the unit ball in  $(n+4A)$ -dimensional Euclidean space.

Similar to [48], Lemma 2.1 implies a local  $f$ -volume doubling property. Indeed, in (2.1), letting  $r_1 = R_1 = 0$ ,  $r_2 = r$  and  $R_2 = 2r$ , from (2.5) we get

$$(2.6) \quad V_f(B_x(2r)) \leq 2^{n+4A} e^{2(n-1+4A)\sqrt{K}r} \cdot V_f(B_x(r))$$

This local  $f$ -volume doubling property is crucial in our proof of Poincaré inequality, Sobolev inequality, mean-value inequality, and Harnack inequality.

From Lemma 2.1, we also have the following,

**Lemma 2.3.** *Let  $(M, g, e^{-f} dv)$  be an  $n$ -dimensional complete noncompact smooth metric measure space. If  $\text{Ric}_f \geq -(n-1)K$  for some constant  $K > 0$ , then*

$$V_f(B_x(r)) \leq \frac{e^{(n-1+4A)\sqrt{K}(d(x,y)+r)}}{r^{n+4A}} V_f(B_y(r)),$$

where  $A = A(y, d(x, y) + r)$ .

*Proof.* We let  $r_1 = 0$ ,  $r_2 = r$ ,  $R_1 = d(x, y) - r$  and  $R_2 = d(x, y) + r$  in Lemma 2.1. Then using (2.5) we have

$$\frac{V_f(B_y(d(x, y) + r)) - V_f(B_y(d(x, y) - r))}{V_f(B_y(r))} \leq \frac{e^{(n-1+4A)\sqrt{K}(d(x, y)+r)}}{r^{n+4A}}.$$

Therefore we get

$$\begin{aligned} V_f(B_x(r)) &\leq V_f(B_y(d(x, y) + r)) - V_f(B_y(d(x, y) - r)) \\ &\leq \frac{e^{(n-1+4A)\sqrt{K}(d(x, y)+r)}}{r^{n+4A}} V_f(B_y(r)). \end{aligned}$$



□

Following the argument of [5] (see also [42] or [34]), applying Lemma 2.1 we get a local Neumann Poincaré inequality on complete smooth metric measure spaces.

**Lemma 2.4.** *Let  $(M, g, e^{-f} dv)$  be an  $n$ -dimensional complete noncompact smooth metric measure space with  $\text{Ric}_f \geq -(n-1)K$  for some constant  $K \geq 0$ . Then,*

$$(2.7) \quad \int_{B_x(r)} |\varphi - \varphi_{B_x(r)}|^2 d\mu \leq c_1 e^{c_2 A + c_3(1+A)\sqrt{K}r} \cdot r^2 \int_{B_x(r)} |\nabla \varphi|^2 d\mu$$

for any  $\varphi \in C^\infty(B_x(r))$ , where  $\varphi_{B_x(r)} = \int_{B_x(r)} \varphi d\mu / \int_{B_x(r)} d\mu$ .

**Remark 2.5.** By Remark 2.2, the coefficient  $c_2 A + c_3(1+A)\sqrt{K}r$  in Lemma 2.4 and all following lemmas, propositions, theorems, and corollaries, can be replaced by  $c_2(A' + \sqrt{K})r + c_3 A' \sqrt{K}r^2$ .

Combining Lemma 2.1 and Lemma 2.4 and the argument of [20] (see also [48]), we obtain a local Sobolev inequality on smooth metric measure spaces.

**Lemma 2.6.** *Let  $(M, g, e^{-f} dv)$  be an  $n$ -dimensional complete noncompact smooth metric measure space with  $\text{Ric}_f \geq -(n-1)K$  for some constant  $K \geq 0$ . Then there exists  $\nu > 2$ , such that*

$$(2.8) \quad \left( \int_{B_x(r)} |\varphi|^{\frac{2\nu}{\nu-2}} d\mu \right)^{\frac{\nu-2}{\nu}} \leq \frac{c_1 e^{c_2 A + c_3(1+A)\sqrt{K}r} \cdot r^2}{V_f(B_x(r))^{\frac{2}{\nu}}} \int_{B_x(r)} (|\nabla \varphi|^2 + r^{-2} |\varphi|^2) d\mu$$

for any  $\varphi \in C^\infty(B_x(r))$ .

Applying Lemma 2.6 we obtain a mean value inequality for solutions to the  $f$ -heat equation, which is similar to Theorem 5.2.9 in [42] (see also [48]).

**Proposition 2.7.** *Let  $(M, g, e^{-f} dv)$  be an  $n$ -dimensional complete noncompact smooth metric measure space. Assume (2.8) holds. Fix  $0 < p < \infty$ . There exist constants  $c_1(n, p, \nu)$ ,  $c_2(n, p, \nu)$  and  $c_3(n, p, \nu)$  such that for any  $s \in \mathbb{R}$  and  $0 < \delta < 1$ , any smooth positive subsolution  $u$  of the  $f$ -heat equation in the cylinder  $Q = B_x(r) \times (s - r^2, s)$  satisfies*

$$\sup_{Q_\delta} \{u^p\} \leq \frac{c_1 e^{c_2 A + c_3(1+A)\sqrt{K}r}}{(1-\delta)^{2+\nu} r^2 V_f(B_x(r))} \cdot \int_Q u^p d\mu dt,$$

where  $Q_\delta = B_x(\delta r) \times (s - \delta r^2, s)$ .

Similar to Proposition 2.7, we have

**Proposition 2.8.** *Let  $(M, g, e^{-f} dv)$  be an  $n$ -dimensional complete noncompact smooth metric measure space. Assume (2.8) holds. Fix  $0 < p_0 < 1 + \nu/2$ . There exist constants  $c_1(n, p_0, \nu)$ ,  $c_2(n, p_0, \nu)$  and  $c_3(n, p_0, \nu)$  such that for any  $s \in \mathbb{R}$ ,  $0 < \delta < 1$ , and  $0 < p \leq p_0$ , any smooth positive supersolution  $u$  of the  $f$ -heat equation in the cylinder  $Q = B_x(r) \times (s - r^2, s)$  satisfies*

$$\|u\|_{p_0, Q'_\delta}^p \leq \left\{ \frac{c_1 e^{c_2 A + c_3(1+A)\sqrt{K}r}}{(1-\delta)^{2+\nu} r^2 V_f(B_x(r))} \right\}^{1-p/p_0} \cdot \|u\|_{p, Q}^p,$$

where  $Q'_\delta := B_x(\delta r) \times (s - r^2, s - (1 - \delta)r^2)$ . On the other hand, for any  $0 < p < \bar{p} < \infty$ , there exist constants  $c_4(n, \bar{p}, \nu)$ ,  $c_5(n, \bar{p}, \nu)$  and  $c_6(n, \bar{p}, \nu)$  such that

$$\sup_{Q'_\delta} \{u^{-p}\} \leq \frac{c_4 e^{c_5 A + c_6(1+A)\sqrt{K}r}}{(1 - \delta)^{2+\nu} r^2 V_f(B_x(r))} \cdot \|u^{-1}\|_{p, Q}^p,$$

where  $\|u\|_{p, Q} = \left( \int_Q |u(x, t)|^p d\mu dt \right)^{1/p}$ .

*Proof of Proposition 2.8.* For any nonnegative test function  $\phi \in C_0^\infty(B)$  and any supersolution of the heat equation, we have

$$\int_B (\phi \partial_t u + \nabla \phi \nabla u) d\mu \geq 0.$$

Let  $\phi = \epsilon q u^{q-1} \psi^2$ ,  $w = u^{q/2}$  for  $-\infty < q \leq p(1 + \nu/2)^{-1} < 1$  and  $q \neq 0$ , where  $\epsilon = 1$  if  $q > 0$  and  $\epsilon = -1$  if  $q < 0$ . We get

$$\epsilon \int_B (\psi^2 \partial_t w^2 + 4(1 - 1/q) \psi^2 |\nabla w|^2 + 4w \psi \langle \nabla w, \nabla \psi \rangle) d\mu \geq 0.$$

When  $q > 0$ . Since

$$2w \psi \langle \nabla w, \nabla \psi \rangle \geq -a^{-2} \psi^2 |\nabla w|^2 - a^2 w^2 |\nabla \psi|^2$$

for any  $a > 0$ , we get

$$- \int_B \psi^2 \partial_t (w^2) d\mu + c_1 \int_B |\nabla(\psi w)|^2 d\mu \leq c_2 \|\nabla \psi\|_\infty^2 \int_{\text{supp}(\psi)} w^2 d\mu,$$

where  $c_1$  and  $c_2$  depend only on  $q$ . Multiplying a nonnegative smooth function  $\lambda(t)$ , we have

$$-\partial_t \int_B \lambda^2 \psi^2 w^2 d\mu + c_1 \lambda^2 \int_B |\nabla(\psi w)|^2 d\mu \leq c_3 \lambda (\lambda \|\nabla \psi\|_\infty^2 + \|\psi \lambda'\|_\infty) \int_B w^2 d\mu.$$

Choose  $\psi$  and  $\lambda$  such that

$$0 \leq \psi \leq 1, \text{ supp } \psi \subset \sigma B, \psi = 1 \text{ in } \sigma' B, |\nabla \psi| \leq (\kappa r)^{-1},$$

$$0 \leq \lambda \leq 1, \lambda = 1 \text{ in } (-\infty, s - \sigma r^2], \lambda = 0 \text{ in } [s - \sigma' r^2, \infty), |\lambda'| \leq (\kappa r^2)^{-1},$$

where  $0 < \sigma' < \sigma < 1$ ,  $\kappa = \sigma - \sigma'$ . Let  $I_\sigma = [s - \sigma r^2, s]$ , and integrate the above inequality on  $[s - r^2, t]$  for  $t \in I_{\sigma'}$ . We get

$$\sup_{I_{\sigma'}} \int_{\sigma' B} w^2 d\mu + c_1 \int_{Q_{\sigma'}} |\nabla w|^2 d\mu dt \leq c_4 (\kappa r)^{-2} \int_{Q_\sigma} w^2 d\mu dt.$$

By Hölder inequality and Proposition 2.6, for any  $\phi \in C_0^\infty(B)$ , we get

$$\begin{aligned} \int_B \phi^{2(1+2/\nu)} d\mu &\leq \left( \int_B \phi^{2\nu/(\nu-2)} d\mu \right)^{(\nu-2)/\nu} \left( \int_B \phi^2 d\mu \right)^{2/\nu} \\ &\leq C(B) \left( \int_B (|\nabla \phi|^2 + r^{-2} \phi^2) d\mu \right) \left( \int_B \phi^2 d\mu \right)^{2/\nu}, \end{aligned}$$

where  $C(B) := c_1 e^{c_2 A + c_3(1+A)\sqrt{K}r} r^2 V_f^{-2/\nu}$ . Therefore

$$(2.9) \quad \int_{Q_{\sigma'}} u^{q\theta} d\mu dt \leq c_3 C(B) \left( (r\kappa)^{-2} \int_{Q_\sigma} u^q d\mu dt \right)^\theta,$$

where  $\theta = 1 + 2/\nu$ . Let  $p_i = p_0\theta^{-i}$ , notice that by Hölder inequality, for any  $p_i < p = \eta p_i + (1 - \eta)p_{i-1} \leq p_{i-1}$  with  $0 \leq \eta < 1$ ,

$$\|u\|_p^p \leq \|u\|_{p_i}^{\eta p_i} \|u\|_{p_{i-1}}^{(1-\eta)p_{i-1}},$$

so it suffices to prove the estimate for all  $p_i$ .

Fix  $i$ , and let  $q_j = p_i\theta^j$ ,  $1 \leq j \leq i-1$ , so  $0 < q_j < p_0(1 + \nu/2)^{-1}$ . Let  $\sigma_0 = 1$ ,  $\sigma_i = \sigma_{i-1} - \kappa_i$ , where  $\kappa_i = (1 - \delta)2^{-i}$ , so  $\sigma_i = 1 - (1 - \delta)\sum_1^i 2^{-j} > \delta$ . Plugging into inequality (2.9), we get

$$\int_{Q'_{\sigma_j}} u^{q_0\theta^j} d\mu dt \leq c_4^j C(B) \left( (1 - \delta)^{-2} r^{-2} \int_{Q'_{\sigma_{j-1}}} u^{q_0\theta^{j-1}} d\mu dt \right)^\theta,$$

for  $1 \leq j \leq i$ . Therefore

$$\int_{Q'_{\sigma_i}} u^{p_0} d\mu dt \leq c_4^{\sum_{j=1}^i (i-j)\theta^{j+1}} C(B)^{\sum_{j=1}^i \theta^j} [(1 - \delta)r]^{-2\sum_{j=1}^i \theta^{j+1}} \left( \int_Q u^{p_i} d\mu dt \right)^{\theta^i},$$

where the summation is taken from 0 to  $i-1$ . Therefore we obtain

$$\left( \int_{Q'_{\sigma_i}} u^{p_0} d\mu dt \right)^{p_i/p_0} \leq [c_5(1 - \delta)^{-2-\nu} E(B)]^{1-p_i/p_0} \left( \int_Q u^{p_i} d\mu dt \right),$$

where  $E(B) = C(B)^{\nu/2} r^{-2-\nu}$ .

When  $q < 0$ . We get

$$\int_B (\psi^2 \partial_t w^2 + 4(1 - 1/q)\psi^2 |\nabla w|^2 + 4w\psi \langle \nabla w, \nabla \psi \rangle) d\mu \leq 0.$$

Applying the mean value inequality to the last term, we get similarly

$$\int_B \psi^2 \partial_t (w^2) d\mu + c_6 \int_B |\nabla(\psi w)|^2 e^{-f} dv \leq c_7 \|\nabla \psi\|_\infty^2 \int_{\text{supp}(\psi)} w^2 d\mu.$$

By the above argument, we can obtain

$$\int_{Q_{\sigma'}} w^{2\theta} d\mu dt \leq c_8 C(B) \left( (r\kappa)^{-2} \int_{Q_\sigma} w^2 d\mu dt \right)^\theta,$$

where  $\theta = 1 + 2/\nu$ . For any  $\alpha > 1$ ,  $v = u^\alpha$  satisfies

$$\partial_t v - \Delta_f v \geq -\frac{\alpha-1}{\alpha} v^{-1} |\nabla v|^2,$$

applying the above argument again, we also have

$$\int_{Q_{\sigma'}} w^{2\alpha\theta} d\mu dt \leq c_9 C(B) \left( (r\kappa)^{-2} \int_{Q_\sigma} w^{2\alpha} d\mu dt \right)^\theta.$$

Let  $\kappa_i = (1 - \delta)2^{-i-1}$ , and  $\sigma_0 = 1$ ,  $\sigma_i = \sigma_{i-1} - \kappa_i = 1 - \sum_1^i \kappa_j$ , and  $\alpha_i = \theta^i$ . We get

$$\begin{aligned} \left( \int_{Q_{\sigma_{i+1}}} w^{2\theta^{i+1}} d\mu dt \right)^{\theta^{-i-1}} &\leq C(B) \left( c_{10}^{i+1} [(1 - \delta)r]^{-2} \int_{Q_{\sigma_i}} w^{2\theta^i} d\mu dt \right)^\theta \\ &\leq C(B)^{\sum_{j=1}^i \theta^{j-1}} c_{10}^{\sum_{j=1}^i (j+1)\theta^{j-1}} [(1 - \delta)r]^{-2\sum_{j=1}^i \theta^{j-1}} \int_Q w^2 d\mu dt, \end{aligned}$$

where the summation is from 1 to  $i + 1$ . Therefore when  $i \rightarrow \infty$ , we get

$$\sup_{Q_\delta} w^2 \leq c_5 C(B)^{\nu/2} [(1 - \delta)r]^{-2-\nu} \|w\|_{2,Q}^2$$

and the conclusion follows.  $\square$

### 3. MOSER'S HARNACK INEQUALITY FOR $f$ -HEAT EQUATION

In this section we prove Moser's Harnack inequalities for the  $f$ -heat equation using Moser iteration, which will lead to the sharp lower bound estimate for the  $f$ -heat kernel in the next section. The arguments mainly follow those in [31, 32, 41, 42], while more delicate analysis is required to get the accurate estimates, which depend on the potential function. Throughout this section, we will use the second  $f$ -volume comparison, i.e., (2.2) in Section 2.

Recall the notations defined in Introduction. for any point  $x \in M$  and  $r > 0$ ,  $s \in \mathbb{R}$ , and  $0 < \varepsilon < \eta < \delta < 1$ , we denote  $B = B_x(r)$ ,  $\delta B = B_x(\delta r)$  and

$$Q = B \times (s - r^2, s), \quad Q_\delta = \delta B \times (s - \delta r^2, s), \quad Q'_\delta = \delta B \times (s - r^2, s - (1 - \delta)r^2),$$

$$Q_- = \delta B \times (s - \delta r^2, s - \eta r^2), \quad Q_+ = \delta B \times (s - \varepsilon r^2, s).$$

With the above notations, we have the main result in this section.

**Theorem 3.1.** *Let  $(M, g, e^{-f} dv)$  be an  $n$ -dimensional complete noncompact smooth metric measure space with  $\text{Ric}_f \geq -(n - 1)K$  for some constant  $K \geq 0$ . For any point  $x \in M$ ,  $r > 0$ , and any parameters  $0 < \varepsilon < \eta < \delta < 1$ , let  $u$  be a smooth solution of the  $f$ -heat equation in  $Q$ , then there exist constants  $c_1$  and  $c_2$  both depending on  $n, \varepsilon, \eta$  and  $\delta$ , such that*

$$\sup_{Q_-} u \leq c_1 e^{c_2(A'^2 + K)r^2} \inf_{Q_+} u,$$

where  $A' = A'(x, r + 1)$ .

**Remark 3.2.** The coefficient in Theorem 3.1 comes from the second volume comparison Lemma 2.1. On the other hand, the first volume comparison in Lemma 2.1 leads to another Harnack inequality,

$$\sup_{Q_-} \{u\} \leq \exp \left\{ c_1 e^{c_2 A} [(1 + A^2)Kr^2 + 1] \right\} \cdot \inf_{Q_+} \{u\}.$$

Since its proof is very similar to that of Theorem 3.1, we omit the proof here.

We first modify the  $f$ -Poincaré inequality (2.7) in Section 2 to a weighted version, which can be derived by adapting a Whitney-type covering argument, see Sections 5.3.3-5.3.5 in [42],

Let  $\xi : [0, \infty) \rightarrow [0, 1]$  be a non-increasing function such that  $\xi(t) = 0$  for  $t > 1$ , and for some positive constant  $\beta$

$$\xi \left( t + \frac{1-t}{2} \right) \geq \beta \xi(t), \quad 1/2 \leq t \leq 1.$$

Let  $\Psi_B(z) := \xi(\rho(x, z)/r)$  for  $z \in B = B(x, r)$  and  $\Psi_B(z) = 0$  for  $z \in M \setminus B$ , we write  $\Psi(z)$  for short. Then

**Lemma 3.3.** *Let  $(M, g, e^{-f} dv)$  be an  $n$ -dimensional complete noncompact smooth metric measure space with  $\text{Ric}_f \geq -(n-1)K$  for some constant  $K \geq 0$ . There exist constants  $c_1(n, \xi)$ ,  $c_2(n)$  and  $c_3(n)$  such that, for any  $B_x(r) \subset M$ , we have*

$$(3.1) \quad \int_{B_x(r)} |\varphi - \varphi_\Psi|^2 \Psi d\mu \leq c_1 e^{c_2(A' + \sqrt{K})r + c_3 A' \sqrt{K} r^2} \cdot r^2 \int_{B_x(r)} |\nabla \varphi|^2 \Psi d\mu$$

for all  $\varphi \in C^\infty(B_x(r))$ , where  $\varphi_\Psi = \int_B \varphi \Psi d\mu / \int_B \Psi d\mu$ .

Secondly, for a positive solution  $u$  to the  $f$ -heat equation, we derive an estimate for the level set of  $\log u$ , the proof of which depends on Lemma 3.3. This inequality is important for the iteration arguments in Lemma 3.5. In the following, we denote  $d\bar{\mu} = d\mu \times dt$  by the natural product measure on  $M \times \mathbb{R}$ .

**Lemma 3.4.** *Let  $(M, g, e^{-f} dv)$  be an  $n$ -dimensional complete noncompact smooth metric measure space. Assume that (2.6) and (2.7) hold in  $B_x(r)$ . Fix  $s \in \mathbb{R}$ ,  $\delta, \tau \in (0, 1)$ . For any smooth positive solution  $u$  of the  $f$ -heat equation in  $Q = B_x(r) \times (s - r^2, s)$ , there exists a constant  $c = c(u)$  depending on  $u$  such that for all  $\lambda > 0$ ,*

$$\begin{aligned} \bar{\mu}(\{(z, t) \in R_+ | \log u < -\lambda - c\}) &\leq C_0 \lambda^{-1}, \\ \bar{\mu}(\{(z, t) \in R_- | \log u > \lambda - c\}) &\leq C_0 \lambda^{-1}, \end{aligned}$$

where  $C_0 = c_1(n, \delta, \tau) e^{c_2(A' + \sqrt{K})r + c_3 A' \sqrt{K} r^2} V_f(B) r^2$ . Here  $R_+ = \delta B \times (s - \tau r^2, s)$  and  $R_- = \delta B \times (s - r^2, s - \tau r^2)$ .

*Proof.* By shrinking the ball  $B$  a little, we can assume that  $u$  is a positive solution in  $B_x(r') \times (s - r^2, s)$  for some  $r' > r$ . Let  $\omega = -\log u$ . Then for any nonnegative function  $\psi \in C_0(B_x(r'))$ , we have

$$\partial_t \int \psi^2 \omega d\mu = - \int \psi^2 u^{-1} \Delta_f u d\mu = \int [-\psi^2 |\nabla \omega|^2 - 2\psi \nabla \omega \cdot \nabla \psi] d\mu.$$

By Cauchy-Schwarz inequality  $2|ab| \leq 1/2a^2 + 2b^2$ , we obtain

$$\partial_t \int \psi^2 \omega d\mu + 1/2 \int |\nabla \omega|^2 d\mu \leq 2 \|\nabla \psi\|_\infty^2 V_f(\text{supp}(\psi)).$$

Fix  $0 < \delta < 1$  and define function  $\xi$  such that  $\xi = 1$  on  $[0, \delta]$ ,  $\xi(t) = \frac{1-t}{1-\delta}$  on  $[\delta, 1]$  and  $\xi = 0$  on  $[1, \infty)$ . We set  $\Psi = \xi(\rho(x, \cdot)/r)$ . Clearly, we can apply the above to  $\psi = \Psi$ . Then Lemma 3.3 can be applied with  $\Psi^2$  as a weight function. Thus, we have

$$\int |\nabla \omega|^2 \Psi^2 d\mu \geq \left( c_\delta r^2 e^{c_2(A' + \sqrt{K})r + c_3 A' \sqrt{K} r^2} \right)^{-1} \int |\omega - W|^2 \Psi^2 d\mu,$$

where  $W := \int \Psi^2 \omega d\mu / \int \Psi^2 d\mu$ . Noticing that  $\int \Psi^2$  is comparable to  $V_f$ , so

$$\partial_t W + C_1^{-1} \int_{\delta B} |\omega - W|^2 \leq C_2,$$

where  $C_1 = C(\delta, \tau) e^{c_2(A' + \sqrt{K})r + c_3 A' \sqrt{K} r^2} r^2 V_f$  and  $C_2 = C(\delta, \tau) r^{-2}$ . Letting  $s' = s - \tau r^2$ , the above inequality can be written as

$$\partial_t \bar{W} + C_1^{-1} \int_{\delta B} |\bar{\omega} - \bar{W}|^2 \leq 0,$$

where  $\bar{\omega}(z, t) = \omega(z, t) - C_2(t - s')$  and  $\bar{W}(z, t) = W(z, t) - C_2(t - s')$ .

Now we set

$$c = W(s') = \overline{W}(s'),$$

and for  $\lambda > 0$ ,  $s - r^2 < t < s$ , we define two sets

$$\Omega_t^+(\lambda) = \{z \in \delta B, \bar{\omega}(z, t) > c + \lambda\} \quad \text{and} \quad \Omega_t^-(\lambda) = \{z \in \delta B, \bar{\omega}(z, t) < c - \lambda\}.$$

Then if  $t > s'$ , we have

$$\bar{\omega}(z, t) - \overline{W}(t) \geq \lambda + c - \overline{W}(t) > \lambda$$

in  $\Omega_t^+(\lambda)$ , since  $c = \overline{W}(s')$  and  $\partial_t \overline{W} \leq 0$ . Similarly, if  $t < s'$ , then we have

$$\bar{\omega}(z, t) - \overline{W}(t) \leq -\lambda + c - \overline{W}(s') < -\lambda$$

in  $\Omega_t^-(\lambda)$ . Hence, if  $t > s'$ , we obtain

$$\partial_t \overline{W}(t) + C_1^{-1} |\lambda + c - \overline{W}(t)|^2 \mu(\Omega_t^+(\lambda)) \leq 0$$

and namely,

$$-C_1 \partial_t (|\lambda + c - \overline{W}(t)|^{-1}) \geq \mu(\Omega_t^+(\lambda)).$$

Integrating from  $s'$  to  $s$ ,

$$\bar{\mu}(\{(z, t) \in R_+, \bar{\omega} > c + \lambda\}) \leq C_1 \lambda^{-1}.$$

Recalling that  $-\log u = \omega = \bar{\omega} + C_2(t - s')$ , hence

$$\bar{\mu}(\{(z, t) \in R_+, \log u < -\lambda - c\}) \leq (\max\{C_1, C_2 r^4 V_f\}) \lambda^{-1}.$$

This gives the first estimate of the lemma. The second estimate follows from a similar argument by working with  $\Omega_t^-$  and  $t < s'$ .  $\square$

Thirdly, in order to finish the proof of Theorem 3.1, we need the following elementary lemma. This is in fact an iterated procedure. We let  $R_\sigma$ ,  $0 < \sigma \leq 1$  be a collection of subset for some space-time endowed with the measure  $d\bar{\mu}$  such that  $R_{\sigma'} \subset R_\sigma$  if  $\sigma' \leq \sigma$ . Indeed,  $R_\sigma$  will be one of the collections  $Q_\delta$  or  $Q'_\delta$ .

**Lemma 3.5.** *Let  $\gamma$ ,  $C$ ,  $1/2 \leq \delta < 1$ ,  $p_1 < p_0 \leq \infty$  be positive constants, and let  $\varphi$  be a positive smooth function on  $R_1$  such that*

$$\|\varphi\|_{p_0, R_{\sigma'}} \leq \{C(\sigma - \sigma')^{-\gamma} V_f^{-1}(R_1)\}^{1/p-1/p_0} \|\varphi\|_{p, R_\sigma}$$

*for all  $\sigma, \sigma'$ ,  $p$  satisfying  $1/2 \leq \delta \leq \sigma' < \sigma \leq 1$  and  $0 < p \leq p_1 < p_0$ . Besides, if  $\varphi$  also satisfies*

$$\text{Vol}_f(\{z \in R_1, \ln \varphi > \lambda\}) \leq C V_f(R_1) \lambda^{-1}$$

*for all  $\lambda > 0$ , then we have*

$$\|\varphi\|_{p_0, R_\delta} \leq (V_f(R_1))^{1/p_0} e^{C_1(1+C^3)},$$

*where  $C_1$  depends only on  $\gamma$ ,  $\delta$  and a positive lower bound on  $1/p_1 - 1/p_0$ .*

*Proof.* Without loss of generality we may assume that  $\text{Vol}_f(R_1) = 1$ . Let

$$\zeta = \zeta(\sigma) := \ln(\|\varphi\|_{p_0, R_\sigma}), \quad \delta \leq \sigma < 1.$$

We divide  $R_\sigma$  into two sets:  $\{\ln \varphi > \zeta/2\}$  and  $\{\ln \varphi \leq \zeta/2\}$ . Then

$$\begin{aligned} \|\varphi\|_{p, R_\sigma} &\leq \|\varphi\|_{p_0, R_\sigma} \cdot V_f(\{z \in R_\sigma, \ln \varphi > \zeta/2\})^{1/p-1/p_0} + e^{\zeta/2} \\ &\leq e^\zeta \left( \frac{2C}{\zeta} \right)^{1/p-1/p_0} + e^{\zeta/2}, \end{aligned}$$

where  $p < p_0$ . Here in the first inequality we used the Hölder inequality, and in the second inequality we used the second assumption of lemma. In the following we want to choose  $p$  such that the last two terms in above are equal, and  $0 < p \leq p_1$ . This is possible if

$$(1/p - 1/p_0)^{-1} = (2/\zeta) \ln \left( \frac{\zeta}{2C} \right) \leq (1/p_1 - 1/p_0)^{-1}$$

and the last inequality is satisfied as long as

$$\zeta \geq C_2 C,$$

where  $C_2$  depends only on a positive lower bound on  $1/p_1 - 1/p_0$ . Now we assume  $p$  and  $\zeta$  have been chosen as above. Then we obtain

$$\|\varphi\|_{p, R_\sigma} \leq 2e^{\zeta/2}.$$

Using the first assumption of the lemma and the definition of  $\kappa$ , we have

$$\begin{aligned} \kappa(\sigma') &\leq \ln \left\{ 2 (C(\sigma - \sigma')^{-\gamma})^{1/p-1/p_0} e^{\zeta/2} \right\} \\ &= (1/p - 1/p_0) \ln[C(\sigma - \sigma')^{-\gamma}] + \ln 2 + \zeta/2 \end{aligned}$$

for any  $\delta \leq \sigma' < \sigma \leq 1$ . According to our choice of  $p$  above, we get

$$\kappa(\sigma') \leq \frac{\zeta}{2} \left\{ \frac{\ln[C(\sigma - \sigma')^{-\gamma}]}{\ln(\zeta/C)} + \frac{2 \ln 2}{\zeta} + 1 \right\}.$$

Here, on one hand, if we choose

$$\zeta \geq 16C^3(\sigma - \sigma')^{-2\gamma} + 8 \ln 2,$$

then the above inequality becomes

$$\zeta(\sigma') \leq \frac{3}{4}\zeta.$$

On the other hand, if the assumption of  $\kappa$  above is not satisfied, we can have

$$\zeta(\sigma') \leq \zeta(\sigma) \leq C_2 C + 16C^3(\sigma - \sigma')^{-2\gamma} + 8 \ln 2.$$

Therefore, in any case

$$\zeta(\sigma') \leq \frac{3}{4}\zeta(\sigma) + C_3(1 + C^3)(\sigma - \sigma')^{-2\gamma}.$$

for any  $\delta \leq \sigma' < \sigma \leq 1$ , where  $C_3 = C_2 + 16 + 8 \ln 2$ . From this, an routine iteration (see [32], page 733) yields

$$\zeta(\delta) \leq C_4(1 - \delta)^{-2\gamma}(1 + C^3),$$

where  $C_4$  depends on  $C_3$  and  $\gamma$ . This completes the proof of the lemma.  $\square$

Now, applying Lemma 3.4, Lemma 3.5 and Proposition 2.8, we get the following Harnack inequality.

**Theorem 3.6.** *Let  $(M, g, e^{-f} dv)$  be an  $n$ -dimensional complete noncompact smooth metric measure space. Assume that (2.6) and (2.7) hold in  $B_x(r)$ . Fix  $\tau \in (0, 1)$  and  $0 < p_0 < 1 + \nu/2$ . For any  $s \in \mathbb{R}$  and  $0 < \varepsilon < \eta < \delta < 1$ , any smooth positive solution  $u$  of the  $f$ -heat equation in the cylinder  $Q = B_x(r) \times (s - r^2, s)$  satisfies*

$$\|u\|_{p_0, Q_-} \leq (r^2 V_f)^{\frac{1}{p_0}} e^{c_1 F(r)} \inf_{Q_+} u,$$

where  $c_1 = c_1(n, \varepsilon, \eta, \delta, p_0)$  and  $F(r) = e^{c_2(A' + \sqrt{K})r + c_3 A' \sqrt{K} r^2}$ ,  $A' = A'(x, r)$ . Hence we have

$$\sup_{Q_-} u \leq e^{c_4 F(r)} \inf_{Q_+} u,$$

where  $c_4 = c_4(n, \varepsilon, \eta, \delta)$ .

*Proof of Theorem 3.6.* We let  $u$  be a positive solution to the  $f$ -heat equation in  $Q$ . Let also  $\delta, \tau \in (0, 1)$  be fixed. Using Proposition 2.8 and Lemma 3.4, we see that Lemma 3.5 can be applied to  $e^c u$  (resp.  $e^{-c} u^{-1}$ ), where  $c = c(u)$  is defined as in Lemma 3.4, with

$$R_\sigma = \sigma \delta B \times (s - r^2, s - \tau r^2 - \sigma \tau r^2) \quad (\text{resp. } R_\sigma = \sigma \delta B \times (s - \sigma \tau r^2, s))$$

and  $0 < p_1 = p_0/2 < p_0 < 1 + \nu/2$  (resp.  $0 < p_1 = 1 < p_0 = \infty$ ). Hence for any  $0 < \varepsilon < \eta < \delta < 1$  and  $Q_-, Q_+$  as defined as above, we have

$$e^c \|u\|_{p_0, Q_-} \leq (r^2 V_f)^{1/p_0} e^{c_1 F(r)}$$

and

$$e^{-c} \sup_{Q_+} \{u^{-1}\} \leq e^{c_4 F(r)},$$

where  $c_1 = c_1(n, \varepsilon, \eta, \delta, p_0)$ ,  $c_4 = c_4(n, \varepsilon, \eta, \delta)$  and  $F(r) = e^{c_2(A' + \sqrt{K})r + c_3 A' \sqrt{K} r^2}$ . The theorem follows from this and Proposition 2.7.  $\square$

Finally, we finish the proof of Theorem 3.1 by applying the standard chain argument to Theorem 3.6.

*Proof of theorem 3.1.* Let  $(t_-, x_-) \in Q_-$ ,  $(t_+, x_+) \in Q_+$ , and let  $\tau = t_+ - t_-$ . Notice that  $\tau \sim r^2$  and  $d = d(x_-, x_+) < r$ . Let  $t_i = t_- + \frac{i\tau}{N}$  and  $x_i \in \frac{1+\delta}{2}B$  for  $0 \leq i \leq N$ , such that  $x_0 = x_-$ ,  $x_N = x_+$ , and  $d(x_i, x_{i+1}) \leq C_\delta \frac{d}{N}$ . Choose  $N$  to be the smallest number such that

$$N \geq C_{\varepsilon, \eta, \delta} (A' + \sqrt{K})^2 r^2,$$

where  $A' = A'(x, r+1)$ , applying Theorem 3.6 with  $r' = (\frac{\tau}{N})^{\frac{1}{2}}$ , then we have

$$\begin{aligned} u(t_-, x_-) &\leq e^{c_4 F(r')(N+1)} u(t_+, x_+) \\ &\leq e^{c_4 F\left(\frac{1}{C(A' + \sqrt{K})}\right)(N+1)} u(t_+, x_+) \\ &\leq \exp \left[ c(A' + \sqrt{K})^2 r^2 + c \right] u(t_+, x_+), \end{aligned}$$

where  $c$  depends on  $n, \varepsilon, \eta$  and  $\delta$ . This finishes the proof of Theorem 3.1.  $\square$

#### 4. GAUSSIAN UPPER AND LOWER BOUNDS OF THE $f$ -HEAT KERNEL

In this section, following the arguments in [42], we derive Gaussian upper and lower bounds for the  $f$ -heat kernel on smooth metric measure spaces. The upper bound estimate follows from the  $f$ -mean value inequality in Proposition 2.7 and a weighted version of Davies integral estimate (see [48]). The lower bound estimate follows from the local Harnack inequality in Section 3.

Let us first state the weighted Davies integral estimate, see [48] for the proof,



**Lemma 4.1.** *Let  $(M, g, e^{-f} dv)$  be an  $n$ -dimensional complete smooth metric measure space. Let  $\lambda_1(M) \geq 0$  be the bottom of the  $L_f^2$ -spectrum of the  $f$ -Laplacian on  $M$ . Assume that  $B_1$  and  $B_2$  are bounded subsets of  $M$ . Then*

$$(4.1) \quad \int_{B_1} \int_{B_2} H(x, y, t) d\mu(x) d\mu(y) \leq V_f^{1/2}(B_1) V_f^{1/2}(B_2) \exp \left( -\lambda_1(M)t - \frac{d^2(B_1, B_2)}{4t} \right),$$

where  $d(B_1, B_2)$  denotes the distance between the sets  $B_1$  and  $B_2$ .

*Proof of upper bound estimate in Theorem 1.1.* For  $x \in B_o(R/2)$ , denote  $u(y, s) = H(x, y, s)$ . Assume  $t \geq r_2^2$ , applying Proposition 2.7 to  $u$ , we have

$$(4.2) \quad \begin{aligned} \sup_{(y,s) \in Q_\delta} H(x, y, s) &\leq \frac{c_1 e^{c_2 A + c_3(1+A)\sqrt{K}r_2}}{r_2^2 V_f(B_2)} \cdot \int_{t-1/4r_2^2}^t \int_{B_2} H(x, \zeta, s) d\mu(\zeta) ds \\ &= \frac{c_1 e^{c_2 A + c_3(1+A)\sqrt{K}r_2}}{4V_f(B_2)} \cdot \int_{B_2} H(x, \zeta, s') d\mu(\zeta) \end{aligned}$$

for some  $s' \in (t - 1/4r_2^2, t)$ , where  $Q_\delta = B_y(\delta r_2) \times (t - \delta r_2^2, t)$  with  $0 < \delta < 1/4$ , and  $B_2 = B_y(r_2) \subset B_o(R)$  for  $y \in B_o(R/2)$ ,  $A = A(x, R) \leq A(o, 2R)$ . Applying Proposition 2.7 and the same argument to the positive solution

$$v(x, s) = \int_{B_2} H(x, \zeta, s) d\mu(\zeta)$$

of the  $f$ -heat equation, for the variable  $x$  with  $t \geq r_1^2$ , we also get

$$(4.3) \quad \begin{aligned} \sup_{(x,s) \in \bar{Q}_\delta} \int_{B_2} H(x, \zeta, s) d\mu(\zeta) &\leq \frac{c_1 e^{c_2 A + c_3(1+A)\sqrt{K}r_1}}{r_1^2 V_f(B_1)} \cdot \int_{t-1/4r_1^2}^t \int_{B_1} \int_{B_2} H(\xi, \zeta, s) d\mu(\zeta) d\mu(\xi) ds \\ &= \frac{c_1 e^{c_2 A + c_3(1+A)\sqrt{K}r_1}}{4V_f(B_1)} \cdot \int_{B_1} \int_{B_2} H(\xi, \zeta, s'') d\mu(\zeta) d\mu(\xi) \end{aligned}$$

for some  $s'' \in (t - 1/4r_1^2, t)$ , where  $\bar{Q}_\delta = B_x(\delta r_1) \times (t - \delta r_1^2, t)$  with  $0 < \delta < 1/4$ , and  $B_1 = B_x(r_1) \subset B_o(R)$  for  $x \in B_o(R/2)$ . Now letting  $r_1 = r_2 = \sqrt{t}$  and combining (4.2) with (4.3), the smooth  $f$ -heat kernel satisfies

$$(4.4) \quad H(x, y, t) \leq \frac{c_1 e^{c_2 A + c_3(1+A)\sqrt{K}t}}{V_f(B_1) V_f(B_2)} \cdot \int_{B_1} \int_{B_2} H(\xi, \zeta, s'') d\mu(\zeta) d\mu(\xi)$$

for all  $x, y \in B_o(R/2)$  and  $0 < t < R^2/4$ . Using Lemma 4.1 and noticing that  $s'' \in (\frac{3}{4}t, t)$ , then (4.4) becomes

$$(4.5) \quad H(x, y, t) \leq \frac{c_1 e^{c_2 A + c_3(1+A)\sqrt{K}t}}{V_f(B_x(\sqrt{t}))^{1/2} V_f(B_y(\sqrt{t}))^{1/2}} \times \exp \left( -\frac{3}{4}\lambda_1 t - \frac{d^2(B_1, B_2)}{4t} \right)$$

for all  $x, y \in B_o(R/2)$  and  $0 < t < R^2/4$ . Notice that if  $d(x, y) \leq 2\sqrt{t}$ , then  $d(B_x(\sqrt{t}), B_y(\sqrt{t})) = 0$  and hence

$$-\frac{d^2(B_x(\sqrt{t}), B_y(\sqrt{t}))}{4t} = 0 \leq 1 - \frac{d^2(x, y)}{4t},$$

and if  $d(x, y) > 2\sqrt{t}$ , then  $d(B_x(\sqrt{t}), B_y(\sqrt{t})) = d(x, y) - 2\sqrt{t}$ , and hence

$$-\frac{d^2(B_x(\sqrt{t}), B_y(\sqrt{t}))}{4t} = -\frac{(d(x, y) - 2\sqrt{t})^2}{4t} \leq -\frac{d^2(x, y)}{4(1+\epsilon)t} + C(\epsilon)$$

for some constant  $C(\epsilon)$ , where  $\epsilon > 0$ . Here if  $\epsilon \rightarrow 0$ , then the constant  $C(\epsilon) \rightarrow \infty$ . Therefore in any case, (4.5) becomes

$$H(x, y, t) \leq \frac{C(\epsilon)e^{c_2 A + c_3(1+A)\sqrt{K}t}}{V_f(B_x(\sqrt{t}))^{1/2}V_f(B_y(\sqrt{t}))^{1/2}} \times \exp\left(-\frac{3}{4}\lambda_1 t - \frac{d^2(x, y)}{4(1+\epsilon)t}\right)$$

for all  $x, y \in B_o(\frac{1}{2}R)$  and  $0 < t < R^2/4$ .  $\square$

Moreover, in Theorem 1.1, if  $K > 0$ . According to Lemma 2.3, we know that

$$V_f(B_x(\sqrt{t})) \leq \frac{e^{(n-1+4A)\sqrt{K}(d(x,y)+\sqrt{t})}}{t^{n/2+2A}} V_f(B_y(\sqrt{t}))$$

for all  $x, y \in B_o(\frac{1}{4}R)$  and  $0 < t < R^2/4$ . Substituting this into Theorem 1.1 yields the following result.

**Corollary 4.2.** *Let  $(M, g, e^{-f} dv)$  be an  $n$ -dimensional complete noncompact smooth metric measure space with  $\text{Ric}_f \geq -(n-1)K$  for some constant  $K > 0$ . For any point  $o \in M$ ,  $R > 0$ ,  $\epsilon > 0$ , there exist constants  $c_1(n, \epsilon)$ ,  $c_2(n)$  and  $c_3(n)$ , such that*

$$(4.6) \quad H(x, y, t) \leq \frac{c_1 e^{c_2 A + c_3(1+A)\sqrt{K}(d(x,y)+\sqrt{t})}}{V_f(B_x(\sqrt{t})) t^{n/4+A}} \times \exp\left(-\frac{d^2(x, y)}{(4+\epsilon)t}\right)$$

for all  $x, y \in B_o(\frac{1}{4}R)$  and  $0 < t < R^2/4$ . Here  $\lim_{\epsilon \rightarrow 0} c_1(n, \epsilon) = \infty$ .

When  $K = 0$ , see the estimate in [48].

Next we derive the lower bound estimate. First, from the Harnack inequality in Theorem 3.1 we get the following estimate,

**Proposition 4.3.** *Under the same assumptions of Theorem 3.1, there exists a constant  $c(n)$  such that, for any two positive solutions  $u(x, s)$  and  $u(y, t)$  of the  $f$ -heat equation in  $B_o(R/2) \times (0, T)$ ,  $0 < s < t < T$ ,*

$$\ln\left(\frac{u(x, s)}{u(y, t)}\right) \leq c(n) \left[ \left( A^2 + K + \frac{1}{R^2} + \frac{1}{s} \right) (t - s) + \frac{d^2(x, y)}{t - s} \right].$$

*Proof.* Let  $u(x, s)$  and  $u(y, t)$  be two positive solutions to the  $f$ -heat equation in  $B_o(\delta R) \times (0, T)$ , where  $x, y \in B_o(\delta R)$  and  $0 < s < t < T$ . Let  $N$  be an integer, which will be chosen later. We set  $t_i = s + i(t - s)/N$ . We remark that it is possible to find a sequence of points  $x_i \in \frac{1+\delta}{2}B$  such that  $x_0 = x$ ,  $x_N = y$  and  $Nd(x_i, x_{i+1}) \geq C_\delta d(x, y)$ . Now we choose  $N$  to be the smallest integer such that

$$\tau/N \leq s/2, \quad \tau/N \leq C_\delta^{-1} R^2, \quad \tau = t - s$$

and if  $d(x, y)^2 \geq \tau$ ,

$$\tau/N \geq d(x, y)^2/N^2.$$

Under the above conditions, we choose

$$N = c_\delta \left( \frac{\tau}{R^2} + \frac{\tau}{s} + \frac{d(x, y)^2}{\tau} \right).$$

Now we apply Theorem 3.1 to compare  $u(x_i, t_i)$  with  $u(x_{i+1}, t_{i+1})$  with  $r' = (\tau/N)^{1/2}$ . Therefore

$$\begin{aligned} \ln \left( \frac{u(x, s)}{u(y, t)} \right) &\leq c_1 \left[ (A'^2 + K) \frac{\tau}{N} + 1 \right] \cdot N \\ &\leq c'_1 \left[ (A'^2 + K) \tau + \frac{\tau}{R^2} + \frac{\tau}{s} + \frac{d(x, y)^2}{\tau} \right], \end{aligned}$$

where  $c'_1$  depends on  $n$  and  $\delta$ , and  $\tau = t - s$ . Then the conclusion follows by letting  $\delta = 1/2$ .  $\square$

From Corollary 4.3, we get the following lower bound for  $f$ -heat kernel,

**Theorem 4.4.** *Let  $(M, g, e^{-f} dv)$  be an  $n$ -dimensional complete noncompact smooth metric measure space with  $\text{Ric}_f \geq -(n-1)K$  for some constant  $K > 0$ . For any point  $o \in M$  and  $R > 0$ , there exist constants  $c_1(n)$ ,  $c_2(n)$  and  $c_3(n)$  such that*

$$(4.7) \quad H(x, y, t) \geq \frac{c_1 e^{-c_2(A'^2 + K)t}}{V_f(B_x(\sqrt{t}))} \times \exp \left( -\frac{d^2(x, y)}{c_3 t} \right),$$

for all  $x, y \in B_o(\frac{1}{2}R)$  and  $0 < t < R^2/4$ .

*Proof of Theorems 4.4 and the second part of Theorem 1.1.* Let  $u(y, t) = H(x, y, t)$  with  $x$  fixed and  $s = t/2$  in Proposition 4.3 and then we get

$$(4.8) \quad H(x, y, t) \geq H(x, x, t/2) \times \exp \left[ -c_1 \left( (A'^2 + K)t + 1 + \frac{t}{R^2} + \frac{d^2(x, y)}{t} \right) \right]$$

for all  $x, y \in B_o(\frac{1}{2}R)$  and  $0 < t < \infty$ .

In the following we will show that Moser's Harnack inequality leads to a lower bound of the on-diagonal  $f$ -heat kernel  $H(x, x, t)$ . Indeed we define

$$u(y, t) = \begin{cases} P_t \phi(y) & \text{if } t > 0 \\ \phi(y) & \text{if } t \leq 0, \end{cases}$$

where  $P_t = e^{t\Delta_f}$  is the heat semigroup of  $\Delta_f$ , and  $\phi$  is a smooth function such that  $0 \leq \phi \leq 1$ ,  $\phi = 1$  on  $B = B_x(\sqrt{t})$  and  $\phi = 0$  on  $M \setminus 2B$ .

$u(y, t)$  satisfies  $(\partial_t - \Delta_f)u = 0$  on  $B \times (-\infty, \infty)$ . Applying the local Harnack inequality, first to  $u$ , and then to the  $f$ -heat kernel  $(y, s) \rightarrow H(x, y, s)$ , we have

$$\begin{aligned} 1 = u(x, 0) &\leq \exp \{ c_1 [(A'^2 + K)t + 1] \} u(x, t/2) \\ &= \exp \{ c_1 [(A'^2 + K)t + 1] \} \int_{B(x, \sqrt{t})} H(x, y, t/2) \phi(y) d\mu(y) \\ &\leq \exp \{ c_1 [(A'^2 + K)t + 1] \} \int_{B(x, 2\sqrt{t})} H(x, y, t/2) d\mu(y) \\ &\leq \exp \{ 2c_1 [(A'^2 + K)t + 1] \} V_f(B_x(2\sqrt{t})) H(x, x, t). \end{aligned}$$

From this, we have

$$H(x, x, t/2) \geq V_f^{-1}(B_x(\sqrt{2t})) \exp \left[ -c_1 \left( (A'^2 + K)t + 2 \right) \right]$$

for  $0 < \sqrt{t} < R/2$ . Since (2.6) implies

$$V_f(B_x(\sqrt{2t})) \leq V_f(B_x(2\sqrt{t})) \leq c_1 e^{c_2(A' + \sqrt{K})\sqrt{t} + c_3 A' \sqrt{K} t} V_f(B_x(\sqrt{t})),$$

we then obtain

$$H(x, x, t/2) \geq V_f^{-1}(B_x(\sqrt{t}))c_4 \exp \left[ -c_5 \left( (A'^2 + K)t + 1 \right) \right]$$

for  $0 < \sqrt{t} < R/2$ . Plugging this into (4.8) yields (4.7).  $\square$

## 5. $L_f^1$ -LIOUVILLE THEOREM

In this section, inspired by the work of P. Li [23], we prove a Liouville theorem for  $f$ -subharmonic functions, and a uniqueness result for solutions of  $f$ -heat equation, by applying the  $f$ -heat kernel upper bound estimates. Our results not only extend the classical  $L^1$ -Liouville theorems proved by P. Li [23], but also generalize the weighted versions in [27], [47], and [48].

Firstly we prove an  $L_f^1$ -Liouville theorem for  $f$ -harmonic functions when the Bakry-Émery Ricci curvature is bounded below and  $f$  is of linear growth.

**Theorem 5.1.** *Let  $(M, g, e^{-f} dv)$  be an  $n$ -dimensional complete noncompact smooth metric measure space with  $\text{Ric}_f \geq -(n-1)K$  for some constant  $K > 0$ . Assume there exist nonnegative constants  $a$  and  $b$  such that*

$$|f|(x) \leq ar(x) + b \text{ for all } x \in M,$$

*where  $r(x)$  is the geodesic distance function to a fixed point  $o \in M$ . Then any nonnegative  $L_f^1$ -integrable  $f$ -subharmonic function must be identically constant. In particular, any  $L_f^1$ -integrable  $f$ -harmonic function must be identically constant.*

*Sketch proof of Theorem 5.1.* We first show that the assumptions of Theorem 5.1 imply the integration by parts formula

$$\int_M \Delta_f H(x, y, t) h(y) d\mu(y) = \int_M H(x, y, t) \Delta_f h(y) d\mu(y)$$

for any nonnegative  $L_f^1$ -integrable  $f$ -subharmonic function  $h$ . This can be proved by our upper bound of  $f$ -heat kernel in Theorem 1.1. Then following the arguments of [48], applying the regularity theory of  $f$ -harmonic functions, we obtain the  $L_f^1$ -Liouville result. See the proof of Theorem 1.5 in [48] for the details.  $\square$

Now we are ready to check the integration by parts formula, similar to the proof of Theorem 4.3 in [48],

**Proposition 5.2.** *Under the same assumptions of Theorem 5.1, for any nonnegative  $L_f^1$ -integrable  $f$ -subharmonic function  $h$ , we have*

$$\int_M \Delta_f H(x, y, t) h(y) d\mu(y) = \int_M H(x, y, t) \Delta_f h(y) d\mu(y).$$

*Proof of Proposition 5.2.* By the Green formula on  $B_o(R)$ , we have

$$\begin{aligned} & \left| \int_{B_o(R)} \Delta_f H(x, y, t) h(y) d\mu(y) - \int_{B_o(R)} H(x, y, t) \Delta_f h(y) d\mu(y) \right| \\ & \leq \int_{\partial B_o(R)} H(x, y, t) |\nabla h|(y) d\mu_{\sigma, R}(y) + \int_{\partial B_o(R)} |\nabla H|(x, y, t) h(y) d\mu_{\sigma, R}(y), \end{aligned}$$

where  $d\mu_{\sigma, R}$  denotes the weighted area measure induced by  $d\mu$  on  $\partial B_o(R)$ . In the following we will show that the above two boundary integrals vanish as  $R \rightarrow \infty$ .

Consider a large  $R$  and assume  $x \in B_o(R/8)$ . By Proposition 2.7, we have the  $f$ -mean value inequality

$$(5.1) \quad \begin{aligned} \sup_{B_o(R)} h(x) &\leq c_1 e^{c_2(aR+b)+c_3(1+aR+b)\sqrt{K}R} V_f^{-1}(2R) \int_{B_o(2R)} h(y) d\mu(y) \\ &\leq C e^{\alpha(1+K)R^2} V_f^{-1}(2R) \int_{B_o(2R)} h(y) d\mu(y), \end{aligned}$$

where constants  $C$  and  $\alpha$  depend on  $n$ ,  $a$  and  $b$ . Let  $\phi(y) = \phi(r(y))$  be a nonnegative cut-off function satisfying  $0 \leq \phi \leq 1$ ,  $|\nabla \phi| \leq \sqrt{3}$  and  $\phi(r(y)) = 1$  on  $B_o(R+1) \setminus B_o(R)$ ,  $\phi(r(y)) = 1$  on  $B_o(R-1) \cup (M \setminus B_o(R+2))$ . Since  $h$  is  $f$ -subharmonic, by the integration by parts formula and Cauchy-Schwarz inequality, we have

$$\begin{aligned} 0 &\leq \int_M \phi^2 h \Delta_f h d\mu = -2 \int_M \phi h \langle \nabla \phi, \nabla h \rangle d\mu - \int_M \phi^2 |\nabla h|^2 d\mu \\ &\leq 2 \int_M |\nabla \phi|^2 h^2 d\mu - \frac{1}{2} \int_M \phi^2 |\nabla h|^2 d\mu. \end{aligned}$$

Then using the definition of  $\phi$  and (5.1), we have that

$$\begin{aligned} \int_{B_o(R+1) \setminus B_o(R)} |\nabla h|^2 d\mu &\leq 4 \int_M |\nabla \phi|^2 h^2 d\mu \\ &\leq 12 \int_{B_o(R+2)} h^2 d\mu \\ &\leq 12 \sup_{B_o(R+2)} h \cdot \|h\|_{L^1(\mu)} \\ &\leq \frac{C e^{\alpha(1+K)(R+2)^2}}{V_f(2R+4)} \cdot \|h\|_{L^1(\mu)}^2. \end{aligned}$$

On the other hand, the Cauchy-Schwarz inequality also implies

$$\int_{B_o(R+1) \setminus B_o(R)} |\nabla h| d\mu \leq \left( \int_{B_o(R+1) \setminus B_o(R)} |\nabla h|^2 d\mu \right)^{1/2} \cdot [V_f(R+1) \setminus V_f(R)]^{1/2}.$$

Combining the above two inequalities we get

$$(5.2) \quad \int_{B_o(R+1) \setminus B_o(R)} |\nabla h| d\mu \leq C_1 e^{\alpha(1+K)R^2} \cdot \|h\|_{L^1(\mu)},$$

where  $C_1 = C_1(n, a, b, K)$ .

We now estimate the  $f$ -heat kernel  $H(x, y, t)$ . Recall that, by letting  $\epsilon = 1$  in Corollary 4.2, the  $f$ -heat kernel  $H(x, y, t)$  satisfies

$$(5.3) \quad \begin{aligned} H(x, y, t) &\leq \frac{c_1 e^{c_2 A + c_3(1+A)\sqrt{K}(d(x,y)+\sqrt{t})}}{V_f(B_x(\sqrt{t}) t^{n/4+A})} \times \exp\left(-\frac{d^2(x, y)}{5t}\right) \\ &\leq \frac{c_4 e^{c_5 R}}{V_f(B_x(\sqrt{t}) t^{c_7(R+1)})} \exp\left[c_6 \sqrt{K}(1+R)(d(x, y) + \sqrt{t}) - \frac{d^2(x, y)}{5t}\right] \end{aligned}$$

for any  $x, y \in B_o(R/2)$  and  $0 < t < R^2/4$ , where  $c_4, c_5, c_6$  and  $c_7$  are all constants depending only on  $n, a$  and  $b$ . Together with (5.2) we get

$$\begin{aligned} J_1 &:= \int_{B_o(R+1) \setminus B_o(R)} H(x, y, t) |\nabla g|(y) d\mu(y) \\ &\leq \sup_{y \in B_o(R+1) \setminus B_o(R)} H(x, y, t) \cdot \int_{B_o(R+1) \setminus B_o(R)} |\nabla g| d\mu \\ &\leq \frac{C_2 \|g\|_{L^1(\mu)}}{V_f(B_x(\sqrt{t})) t^{c_7(R+2)}} \times \exp \left[ c_5 R - \frac{(R - d(o, x))^2}{5t} + c_9 \sqrt{K} (R+2)(R+1 + d(o, x) + \sqrt{t}) \right], \end{aligned}$$

where  $C_2 = C_2(n, a, b, K)$ . Notice that

$$t^{-c_7(R+2)} = e^{-c_7(R+2) \ln t} \leq e^{c_7(R+2) \frac{1}{t}} \quad \text{when } t \rightarrow 0.$$

Thus, for  $T$  sufficiently small and for all  $t \in (0, T)$  there exists a constant  $\beta > 0$  such that

$$J_1 \leq \frac{C_3 \|g\|_{L^1(\mu)}}{V_f(B_x(\sqrt{t}))} \times \exp \left( -\beta R^2 + c \frac{d^2(o, x)}{t} \right),$$

where  $C_3 = C_3(n, a, b, K)$ . Therefore for all  $t \in (0, T)$  and all  $x \in M$ ,  $J_1 \rightarrow 0$  as  $R \rightarrow \infty$ .

By a similar argument, we can show that

$$\int_{B_o(R+1) \setminus B_o(R)} |\nabla H|(x, y, t) h(y) d\mu \rightarrow 0$$

as  $R \rightarrow \infty$ . We first estimate  $\int_{B_o(R+1) \setminus B_o(R)} |\nabla H|(x, y, t) d\mu$ .

$$\begin{aligned} \int_M \phi^2(y) |\nabla H|^2(x, y, t) d\mu &= -2 \int_M \langle H(x, y, t) \nabla \phi(y), \phi(y) \nabla H(x, y, t) \rangle d\mu \\ &\quad - \int_M \phi^2(y) H(x, y, t) \Delta_f H(x, y, t) d\mu \\ &\leq 2 \int_M |\nabla \phi|^2(y) H^2(x, y, t) d\mu + \frac{1}{2} \int_M \phi^2(y) |\nabla H|^2(x, y, t) d\mu \\ &\quad - \int_M \phi^2(y) H(x, y, t) \Delta_f H(x, y, t) d\mu, \end{aligned}$$

which implies

$$\begin{aligned} &\int_{B_o(R+1) \setminus B_o(R)} |\nabla H|^2 \\ &\leq \int_M \phi^2(y) |\nabla H|^2(x, y, t) \\ (5.4) \quad &\leq 4 \int_M |\nabla \phi|^2 H^2 - 2 \int_M \phi^2 H \Delta_f H \\ &\leq 12 \int_{B_o(R+2) \setminus B_o(R-1)} H^2 + 2 \int_{B_o(R+2) \setminus B_o(R-1)} H |\Delta_f H| \\ &\leq 12 \int_{B_o(R+2) \setminus B_o(R-1)} H^2 + 2 \left( \int_{B_o(R+2) \setminus B_o(R-1)} H^2 \right)^{\frac{1}{2}} \left( \int_M (\Delta_f H)^2 \right)^{\frac{1}{2}}. \end{aligned}$$

Notice that by Theorem 4.1 in [45], if  $\text{Ric}_f \geq -(n-1)K$ , then

$$V_f(B_o(R)) \leq A + B \exp \left[ \frac{(n-1)K}{2} R^2 \right]$$

for all  $R > 1$ , so we have

$$(5.5) \quad \int_1^\infty \frac{R}{\log V_f(B_o(R))} dR = \infty.$$

By Theorem 3.13 in [19],  $(M, g, e^{-f} dv)$  is stochastically complete, i.e.,

$$(5.6) \quad \int_M H(x, y, t) e^{-f} dv(y) = 1.$$

Using (5.3) and (5.6), we get

$$(5.7) \quad \begin{aligned} \int_{B_o(R+2) \setminus B_o(R-1)} H^2(x, y, t) d\mu &\leq \sup_{y \in B_o(R+2) \setminus B_o(R-1)} H(x, y, t) \\ &\leq \frac{c_4}{V_f(B_x(\sqrt{t})) t^{c_7(R+3)}} \times \exp \left[ -\frac{(R-1-d(o, x))^2}{5t} \right] \\ &\quad \times \exp \left[ c_5(R+2) + c_6\sqrt{K}(3+R)(R+2-d(o, x) + \sqrt{t}) \right] \\ &= \frac{c_4}{V_f(B_x(\sqrt{t}))} \times \exp \left[ -\frac{(R-1-d(o, x))^2}{5t} + c_7(R+3) \ln \frac{1}{t} \right] \\ &\quad \times \exp \left[ c_5(R+2) + c_6\sqrt{K}(3+R)(R+2-d(o, x) + \sqrt{t}) \right]. \end{aligned}$$

From (4.7) in [48], there exists a constant  $C > 0$  such that

$$(5.8) \quad \int_M (\Delta_f H)^2(x, y, t) d\mu \leq \frac{C}{t^2} H(x, x, t).$$

Combining (5.4), (5.7) and (5.8), we obtain

$$\begin{aligned} \int_{B_o(R+1) \setminus B_o(R)} |\nabla H|^2 d\mu &\leq C_4 \left[ V_f^{-1} + t^{-1} V_f^{-\frac{1}{2}} H^{\frac{1}{2}}(x, x, t) \right] \\ &\quad \times \exp \left[ -\frac{(R-1-d(o, x))^2}{10t} + c_7(R+3) \ln \frac{1}{t} \right] \\ &\quad \times \exp \left[ c_5 R + c_6\sqrt{K}(3+R)(R+2-d(o, x) + \sqrt{t}) \right] \end{aligned}$$

where  $V_f = V_f(B_x(\sqrt{t}))$  and  $C_4 = C_4(n, a, b)$ . Hence we get

$$(5.9) \quad \begin{aligned} \int_{B_o(R+1) \setminus B_o(R)} |\nabla H| d\mu &\leq [V_f(B_o(R+1)) \setminus V_f(B_o(R))]^{1/2} \times \left[ \int_{B_o(R+1) \setminus B_o(R)} |\nabla H|^2 d\mu \right]^{1/2} \\ &\leq C_4 V_f^{1/2}(B_o(R+1)) \left[ V_f^{-1} + t^{-1} V_f^{-\frac{1}{2}} H^{\frac{1}{2}}(x, x, t) \right]^{1/2} \\ &\quad \times \exp \left[ -\frac{(R-1-d(o, x))^2}{20t} + \frac{c_7}{2}(R+3) \ln \frac{1}{t} \right] \\ &\quad \times \exp \left[ c_5 R + c_6\sqrt{K}(3+R)(R+2-d(o, x) + \sqrt{t}) \right]. \end{aligned}$$

Therefore, by (5.1) and (5.9), we obtain

$$\begin{aligned}
J_2 &= \int_{B_o(R+1) \setminus B_o(R)} |\nabla H(x, y, t)| h(y) d\mu(y) \\
&\leq \sup_{y \in B_o(R+1) \setminus B_o(R)} h(y) \cdot \int_{B_o(R+1) \setminus B_o(R)} |\nabla H(x, y, t)| d\mu(y) \\
&\leq \frac{C_5 \|g\|_{L^1(\mu)}}{V_f^{1/2}(B_o(2R+2))} \cdot \left[ V_f^{-1} + t^{-1} V_f^{-\frac{1}{2}} H^{\frac{1}{2}}(x, x, t) \right]^{1/2} \\
&\quad \times \exp \left[ \alpha(1+K)(R+1)^2 - \frac{(R-1-d(o, x))^2}{20t} + \frac{c_7}{2}(R+3) \ln \frac{1}{t} \right] \\
&\quad \times \exp \left[ c_5 R + c_6 \sqrt{K}(3+R)(R+2-d(o, x) + \sqrt{t}) \right],
\end{aligned}$$

where  $C_5 = C_5(n, a, b)$ . Similar to the case of  $J_1$ , we choose  $T$  sufficiently small, then for all  $t \in (0, T)$  and all  $x \in M$ ,  $J_2 \rightarrow 0$  when  $R \rightarrow \infty$ .

Now by the mean value theorem, for any  $R > 0$  there exists  $\bar{R} \in (R, R+1)$  such that

$$\begin{aligned}
J &= \int_{\partial B_o(\bar{R})} [H(x, y, t) |\nabla h|(y) + |\nabla H|(x, y, t) h(y)] d\mu_{\sigma, \bar{R}}(y) \\
&= \int_{B_o(R+1) \setminus B_o(R)} [H(x, y, t) |\nabla h|(y) + |\nabla H|(x, y, t) h(y)] d\mu(y) \\
&= J_1 + J_2.
\end{aligned}$$

By the above argument, we choose  $T$  sufficiently small, then for all  $t \in (0, T)$  and all  $x \in M$ ,  $J \rightarrow 0$  as  $\bar{R} \rightarrow \infty$ . Therefore Proposition 5.2 holds for  $T$  sufficiently small. Then the semigroup property of the  $f$ -heat equation implies Proposition 5.2 holds for all time  $t > 0$ .  $\square$

Theorem 5.1 leads to a uniqueness property for  $L^1$ -solutions of the  $f$ -heat equation, which generalizes the classical result of P. Li [23]. The proof is very similar to the one in [48], so we omit it.

**Theorem 5.3.** *Under the same assumptions of Theorem 5.1, if  $u(x, t)$  is a non-negative function defined on  $M \times [0, +\infty)$  satisfying*

$$(\partial_t - \Delta_f)u(x, t) \leq 0, \quad \int_M u(x, t) e^{-f} dv < +\infty$$

for all  $t > 0$ , and

$$\lim_{t \rightarrow 0} \int_M u(x, t) e^{-f} dv = 0,$$

then  $u(x, t) \equiv 0$  for all  $x \in M$  and  $t \in (0, +\infty)$ . In particular, any  $L_f^1$ -solution of the  $f$ -heat equation is uniquely determined by its initial data in  $L_f^1$ .

## 6. EIGENVALUE ESTIMATE

In this section we derive eigenvalue estimates of the  $f$ -Laplace operator compact smooth metric measure spaces, using the upper bound estimate of the  $f$ -heat kernel and an argument of Li-Yau [26].

When the Bakry-Emery Ricci curvature is nonnegative, we have



**Theorem 6.1.** *Let  $(M, g, e^{-f} dv)$  be an  $n$ -dimensional closed smooth metric measure space with  $\text{Ric}_f \geq 0$ . Let  $0 = \lambda_0 < \lambda_1 \leq \lambda_2 \leq \dots$  be eigenvalues of the  $f$ -Laplacian. Then there exists a constant  $C$  depending only on  $n$  and  $\max_{x \in M} f(x)$ , such that*

$$\lambda_k \geq \frac{C(k+1)^{2/n}}{d^2}$$

for all  $k \geq 1$ , where  $d$  is the diameter of  $M$ .

*Proof.* Since  $\text{Ric}_f \geq 0$ , from Theorem 1.1, we have

$$(6.1) \quad H(x, x, t) \leq \frac{C}{V_f(B_x(\sqrt{t}))},$$

where  $C$  is a constant depending only on  $n$  and  $B = \max_{x \in M} f(x)$ . Notice that the  $f$ -heat kernel can be written as

$$H(x, y, t) = \sum_{i=0}^{\infty} e^{-\lambda_i t} \varphi_i(x) \varphi_i(y),$$

where  $\varphi_i$  is the eigenfunction of  $\Delta_f$  corresponding to  $\lambda_i$ ,  $\|\varphi_i\|_{L_f^2} = 1$ . By the  $f$ -volume comparison theorem (see Lemma 2.1 in [48]), we get, for any  $t \leq d^2$ ,

$$\frac{V_f(B_x(d))}{V_f(B_x(\sqrt{t}))} \leq e^{4B} \left( \frac{d}{\sqrt{t}} \right)^n$$

Taking the weighted integral on both sides of (6.1), we conclude that

$$\sum_{i=0}^{\infty} e^{-\lambda_i t} \leq C \int_M V_f^{-1}(B_x(\sqrt{t})) d\mu \leq C \int_M p(t) d\mu,$$

where

$$p(t) = \begin{cases} e^{4B} \left( \frac{d}{\sqrt{t}} \right)^n V_f^{-1}(B_x(d)), & \text{if } \sqrt{t} \leq d \\ e^{4B} V_f^{-1}(M), & \text{if } \sqrt{t} > d. \end{cases}$$

which implies that  $(k+1)e^{-\lambda_k t} \leq Cq(t)$  for any  $t > 0$ , that is

$$(6.2) \quad Ce^{\lambda_k t} q(t) \geq (k+1), \quad \text{for any } t > 0,$$

where

$$q(t) = \begin{cases} e^{4B} \left( \frac{d}{\sqrt{t}} \right)^n, & \text{if } \sqrt{t} \leq d \\ e^{4B}, & \text{if } \sqrt{t} > d. \end{cases}$$

It is easy to see that  $e^{\lambda_k t} q(t)$  takes its minimum at  $t_0 = \frac{n}{2\lambda_k}$ . Plugging to (6.2) we get the lower bound for  $\lambda_k$ .  $\square$

Similarly, when the Bakry-Émery Ricci curvature is bounded below, we have a similar estimate. We omit the proof since it is the same as  $\text{Ric}_f \geq 0$  case.

**Theorem 6.2.** *Let  $(M, g, e^{-f} dv)$  be an  $n$ -dimensional closed smooth metric measure space with  $\text{Ric}_f \geq -(n-1)K$  for some constant  $K > 0$ . Let  $0 = \lambda_0 < \lambda_1 \leq \lambda_2 \leq \dots$  be eigenvalues of the  $f$ -Laplacian. Then there exists a constant  $C$  depending only on  $n$  and  $B = \max_{x \in M} f(x)$ , such that*

$$\lambda_k \geq \frac{C}{d^2} \left( \frac{k+1}{\exp(C\sqrt{K}d)} \right)^{\frac{2}{n+4B}}$$

for all  $k \geq 1$ , where  $d$  is the diameter of  $M$ .

### 7. $f$ -GREEN'S FUNCTION ESTIMATE

In this section, we will discuss the Green's function of the  $f$ -Laplacian and  $f$ -parabolicity of smooth metric measure spaces. It was proved by Malgrange [30] that every Riemannian manifold admits a Green's function of Laplacian. Varopoulos [44] proved that a complete manifold  $(M, g)$  has a positive Green's function only if

$$(7.1) \quad \int_1^\infty \frac{t}{V_p(t)} dt < \infty,$$

where  $V_p(t)$  is the volume of the geodesic ball of radius  $t$  with center at  $p$ . For Riemannian manifolds with nonnegative Ricci curvature, Varopoulos [44] and Li-Yau [26] proved (7.1) is the sufficient and necessary condition for the existence of positive Green's function.

On an  $n$ -dimensional complete smooth metric measure space  $(M, g, e^{-f} dv)$ , let  $H(x, y, t)$  be a  $f$ -heat kernel, recall the  $f$ -Green's function

$$G(x, y) = \int_0^\infty H(x, y, t) dt$$

if the integral on the right hand side converges. From the  $f$ -heat kernel estimates, it is easy to get the following two-sided estimates for  $f$ -Green's function, which is similar to Li-Yau estimate [26] of Green's function for Riemannian manifolds with nonnegative Ricci curvature,

**Theorem 7.1.** *Let  $(M, g, e^{-f} dv)$  be an  $n$ -dimensional complete noncompact smooth metric measure space with  $\text{Ric}_f \geq 0$  and  $|f| \leq C$  for some nonnegative constant  $C$ . If  $G(x, y)$  exists, then there exist constants  $c_1$  and  $c_2$  depending only on  $n$  and  $C$ , such that*

$$(7.2) \quad c_1 \int_{r^2}^\infty V_f^{-1}(B_x(\sqrt{t})) dt \leq G(x, y) \leq c_2 \int_{r^2}^\infty V_f^{-1}(B_x(\sqrt{t})) dt,$$

where  $r = r(x, y)$ .

As a corollary, we get a necessary and sufficient condition of the existence of positive  $f$ -Green's function on smooth metric measure spaces with nonnegative Bakry-Emery Ricci curvature and bounded potential function,

**Corollary 7.2.** *Let  $(M, g, e^{-f} dv)$  be an  $n$ -dimensional complete noncompact smooth metric measure space with  $\text{Ric}_f \geq 0$  and  $|f| \leq C$  for some nonnegative constant  $C$ . There exists a positive  $f$ -Green's function  $G(x, y)$  if and only if*

$$\int_1^\infty V_f^{-1}(B_x(\sqrt{t})) dt < \infty.$$

*Proof of Theorem 7.1.* Since  $\text{Ric}_f \geq 0$  and  $|f| \leq C$ , Theorem 1.1 holds for any  $0 < t < \infty$  by letting  $R \rightarrow \infty$ . For the lower bound estimate, we have

$$\begin{aligned} G(x, y) &\geq \int_{r^2}^\infty H(x, y, t) dt \geq c_3(n, C) \int_{r^2}^\infty V_f^{-1}(B_x(\sqrt{t})) \exp\left(\frac{-r^2}{c_4 t}\right) dt \\ &\geq c_5(n, C) \int_{r^2}^\infty V_f^{-1}(B_x(\sqrt{t})) dt. \end{aligned}$$

Hence the left hand side of (7.2) follows.

For the upper bound estimate, it suffices to show that

$$(7.3) \quad \int_0^{r^2} H(x, y, t) dt \leq c_6(n, C) \int_{r^2}^{\infty} V_f^{-1}(B_x(\sqrt{t})) dt.$$

By the definition of  $G$  and Theorem 1.1,

$$\begin{aligned} G(x, y) &= \int_0^{\infty} H(x, y, t) dt = \int_0^{r^2} H(x, y, t) dt + \int_{r^2}^{\infty} H(x, y, t) dt \\ &\leq \int_0^{r^2} H(x, y, t) dt + c_7(n, C) \int_{r^2}^{\infty} V_f^{-1}(B_x(\sqrt{t})) dt \\ &\leq c_8 \int_0^{r^2} V_f^{-1}(B_x(\sqrt{t})) \exp\left(\frac{-r^2}{5t}\right) dt + c_7 \int_{r^2}^{\infty} V_f^{-1}(B_x(\sqrt{t})) dt, \end{aligned}$$

where  $c_7$  and  $c_8$  depend on  $n$  and  $C$ . Letting  $s = r^4/t$ , where  $r^2 < s < \infty$ , we get

$$\int_0^{r^2} V_f^{-1}(B_x(\sqrt{t})) \exp\left(\frac{-r^2}{5t}\right) dt = \int_{r^2}^{\infty} V_f^{-1}\left(B_x\left(\frac{r^2}{\sqrt{s}}\right)\right) \exp\left(\frac{-s}{5r^2}\right) \frac{r^4}{s^2} ds.$$

On the other hand, the  $f$ -volume comparison theorem (see Lemma 2.1 in [48]) gives

$$V_f^{-1}\left(B_x\left(\frac{r^2}{\sqrt{s}}\right)\right) \leq V_f^{-1}(B_x(\sqrt{s})) e^{4C} \left(\frac{s}{r^2}\right)^n.$$

Therefore we get

$$\int_0^{r^2} H(x, y, t) dt \leq c_9(n, C) \int_{r^2}^{\infty} V_f^{-1}(B_x(\sqrt{s})) \left(\frac{s}{r^2}\right)^{n-2} \exp\left(\frac{-s}{5r^2}\right) ds.$$

Since the function  $x^{n-2}e^{-x/5}$  is bound from above, (7.3) follows.  $\square$

Next we discuss  $f$ -nonparabolicity of steady Ricci solitons using a criterion of Li-Tam [24, 25], and the  $f$ -heat kernel for steady Gaussian Ricci soliton. A smooth metric measure space  $(M^n, g, e^{-f} dv)$  is called  $f$ -nonparabolic if it admits a positive  $f$ -Green's function. An end,  $E$ , with respect to a compact subset  $\Omega \subset M$  is an unbounded connected component of  $M$ . When we say that  $E$  is an end, it is implicitly assumed that  $E$  is an end with respect to some compact subset  $\Omega \subset M$ . O. Munteanu and J. Wang [34] proved that if  $\text{Ric}_f \geq 0$ , there exists at most one  $f$ -nonparabolic end on  $(M^n, g, e^{-f} dv)$ .

First we observe that the criterion of Li-Tam [24, 25] can be generalized to smooth metric measure spaces,

**Lemma 7.3.** *Let  $(M^n, g, e^{-f} dv)$  be an  $n$ -dimensional complete smooth metric measure space. There exists an  $f$ -Green's function  $G(x, y)$  which is smooth on  $M \times M \setminus D$ , where  $D = \{(x, x) | x \in M\}$ . Moreover,  $G(x, y)$  can be taken to be positive if and only if there exists a positive nonconstant  $f$ -superharmonic function  $u$  on  $M \setminus B_o(r)$  with the property that*

$$\liminf_{x \rightarrow \infty} u(x) < \inf_{x \in \partial B_o(r)} u(x).$$

*Proof of Theorem 1.8.* Let  $(M, g, f)$  be a nontrivial gradient steady soliton, we have

$$\Delta f + R = 0 \quad \text{and} \quad R + |\nabla f|^2 = a.$$

Chen [10] proved that  $R \geq 0$ , so  $a > 0$ . It was proved in [37, 15] (see also [49]) that  $\liminf R = 0$ , and either  $R > 0$  or  $R \equiv 0$ .

By the Bochner formula, we get

$$\Delta_f R = -2|Ric|^2 \leq 0.$$

If  $R > 0$  on  $M$ , then it is a nonconstant positive  $f$ -superharmonic function, and  $\liminf_{x \rightarrow \infty} R(x) = 0$ . Therefore, by Lemma 7.3, we conclude  $G(x, y)$  is positive.

If  $R \equiv 0$ , then by Proposition 4.3 in [37],  $(M^n, g)$  splits isometrically as  $(N^{n-k} \times \mathbb{R}^k, g_N + g_0)$ , where  $(N^{n-k}, g_N)$  is a Ricci-flat manifold, and  $(\mathbb{R}^k, g_0, f)$  is a steady Gaussian Ricci soliton with  $f = \langle u, x \rangle + v$  for some  $u, v \in \mathbb{R}^n$ . Therefore a  $f$ -Green's function on  $(\mathbb{R}^k, g_0, f)$  is a  $f$ -Green's function on  $(M, g, f)$ .

By [48], for one-dimensional steady Gaussian Ricci soliton, the  $f$ -heat kernel is given by

$$H_{\mathbb{R}}(x, y, t) = \frac{e^{\pm \frac{x+y}{2}} \cdot e^{-t/4}}{(4\pi t)^{1/2}} \times \exp\left(-\frac{|x-y|^2}{4t}\right).$$

for any  $x, y \in \mathbb{R}$  and  $t > 0$ . Therefore for any  $x, y \in \mathbb{R}$ ,

$$G(x, y) = \int_0^\infty H_{\mathbb{R}}(x, y, t) dt < \infty,$$

hence there exists a positive  $f$ -Green function.

For higher dimensional steady Gaussian Ricci soliton  $(\mathbb{R}^k, g_0, f)$ , define

$$H_{\mathbb{R}^k}(x, y, t) = H_{\mathbb{R}}(x_1, y_1, t) \times H_{\mathbb{R}}(x_2, y_2, t) \times \dots \times H_{\mathbb{R}}(x_k, y_k, t),$$

where  $x = (x_1, x_2, \dots, x_k) \in \mathbb{R}^k$ ,  $y = (y_1, y_2, \dots, y_k) \in \mathbb{R}^k$ , and  $H_{\mathbb{R}}(x_i, y_i, t)$  is the  $f$ -heat kernel for  $(\mathbb{R}, g_0, u_i x_i + v_i)$ . It is easy to check that  $H_{\mathbb{R}^k}(x, y, t)$  is an  $f$ -heat kernel on  $(\mathbb{R}^k, g_0, f)$ .

Then for any  $x, y \in \mathbb{R}^k$ ,

$$G(x, y) = \int_0^\infty H_{\mathbb{R}^k}(x, y, t) dt < \infty.$$

Therefore there exists a positive  $f$ -Green function on an  $k$ -dimensional steady Gaussian soliton.  $\square$

## REFERENCES

1. B. Andrews, L. Ni, Eigenvalue comparison on Bakry-Emery manifolds, *Comm. Partial Differential Equations* 37 (2012), no. 11, 2081-2092.
2. D. Bakry, M. Emery, Diffusion hypercontractivites, in: *Séminaire de Probabilités XIX, 1983/1984*, in: *Lecture Notes in Math.*, vol. 1123, Springer-Verlag, Berlin, 1985, pp. 177-206.
3. D. Bakry, Z.-M. Qian, Some new results on eigenvectors via dimension, diameter and Ricci curvature, *Adv. Math.* 155 (2000), 98-153.
4. K. Brighton, A Liouville-type theorem for smooth metric measure spaces, *J. Geome. Anal.* 23 (2013), 562-570.
5. P. Buser, A note on the isoperimetric constant, *Ann. Sci. Ecole Norm. Sup.* 15 (1982), 213-230.
6. H.-D. Cao, Recent progress on Ricci solitons, *Recent advances in geometric analysis*, *Adv. Lect. Math. (ALM)* 11, 1-38, International Press, Somerville, MA 2010.
7. H.-D. Cao, D. Zhou, On complete gradient shrinking Ricci solitons, *J. Diff. Geom.* 85 (2010), 175-186.
8. J. Case, Y.-S. Shu, and G. Wei, Rigidity of quasi-Einstein metrics, *Diff. Geo. Appl.* 29 (2011), 93-100.
9. N. Charalambous, Z. Lu, Heat kernel estimates and the essential spectrum on weighted manifolds, *J. Geome. Anal.* (2013), DOI 10.1007/s12220-013-9438-1

10. B.-L. Chen, Strong uniqueness of the Ricci flow, *J. Diff. Geom.* 82 (2009), 363-382.
11. X. Cheng, D. Zhou, Eigenvalues of the drifted Laplacian on complete metric measure spaces, arXiv: 1305.4116.
12. B. Colbois, A. Soufi, A. Savo, Eigenvalues of the Laplacian on a compact manifold with density, arXiv:1310.1490.
13. X. Dai, C. J. Sung, J. Wang, G. Wei, in preparation.
14. E. B. Davies, Heat kernels and spectral theory, *Cambridge Tracts in Mathematics*, vol. 92, Cambridge U. Press, 1989.
15. M Fernández-López, E. García-Río, Maximum principles and gradient Ricci solitons, *J. Differential Equations* 251 (2011), 73-81.
16. A. Futaki, H. Li, X.-D. Li, On the first eigenvalue of the Witten-Laplacian and the diameter of compact shrinking solitons, *Ann. Glob. Anal. Geom.* 44 (2013), 105-114.
17. A. Grigor'yan, The heat equation on noncompact Riemannian manifolds, (Russian) *Math. Sb.* 182 (1991), 55-87; translation in *Math. USSR Sb.* 72 (1992), 47-77.
18. A. Grigor'yan, Heat kernel and analysis on manifolds. *AMS/IP Studies in Advanced Mathematics*, 47. Amer. Math. Soc., Providence, RI; International Press, Boston, MA, 2009.
19. A. Grigor'yan, Heat kernels on weighted manifolds and applications, *The ubiquitous heat kernel*, *Contemp. Math.*, vol. 398, Amer. Math. Soc., Providence, RI, 2006, pp. 93-191.
20. P. Hajlasz and P. Koskela, Sobolev meets Poincaré, *C. R. Acad. Sci. Paris Sr. I Math.* 320 (1995), 1211-1215.
21. R. Hamilton, The formation of singularities in the Ricci flow, *Surveys in Differential Geom.* 2 (1995), 7-136, International Press.
22. A. Hassannezhad, Eigenvalues of perturbed Laplace operators on compact manifolds. *Pacific J. Math.* 264 (2013), 333-354.
23. P. Li, Uniqueness of  $L^1$  solutions for the Laplace equation and the heat equation on Riemannian manifolds, *J. Diff. Geom.* 20 (1984), 447-457.
24. P. Li, L.-F. Tam, Positive harmonic functions on complete manifolds with non-negative curvature outside a compact set, *Ann. of Math.* 125 (1987), 171-207.
25. P. Li, L.-F. Tam, Symmetric Green's functions on complete manifolds, *Amer. J. Math.* 109 (1987), 1129-1154.
26. P. Li, S.-T. Yau, On the parabolic kernel of the Schrödinger operator, *Acta Math.* 156 (1986), 153-201.
27. X.-D. Li, Liouville theorems for symmetric diffusion operators on complete Riemannian manifolds, *J. Math. Pure. Appl.* 84 (2005), 1295-1361.
28. J. Lott, Some geometric properties of the Bakry-Émery-Ricci tensor, *Comment. Math. Helv.* 78 (2003), 865-883.
29. J. Lott, C. Villani, Ricci curvature for metric-measure spaces via optimal transport, *Ann. of Math.* 169 (2009), 903-991.
30. M. Malgrange, Existence et approximation des solutions des équations aux dérivées partielles et des équations de convolution, *Ann. Inst. Fourier* 6 (1955), 271-355.
31. J. Moser, A Harnack inequality for parabolic differential equations, *Comm. Pure Appl. Math.* 17 (1964), 101-134.
32. J. Moser, On a pointwise estimate for parabolic differential equations, *Comm. Pure Appl. Math.* 24 (1971), 727-740.
33. O. Munteanu, N. Sesum, On gradient Ricci solitons, *J. Geom. Anal.* 23 (2013), 539-561.
34. O. Munteanu, J. Wang, Smooth metric measure spaces with nonnegative curvature, *Comm. Anal. Geom.* 19 (2011), 451-486.
35. O. Munteanu, J. Wang, Analysis of weighted Laplacian and applications to Ricci solitons, *Comm. Anal. Geom.* 20 (2012), 55-94.
36. O. Munteanu, J. Wang, Geometry of manifolds with densities, *Adv. Math.* 259 (2014), 269-305.
37. P. Petersen, W. Wylie, Rigidity of gradient Ricci solitons, *Pacific J. Math.* 241 (2009), 329-345.
38. S. Pigola, M. Rimoldi, A.G. Setti, Remarks on non-compact gradient Ricci solitons, *Math. Z.* 268 (2011), 777-790.
39. Z. Qian, Estimates for Weighted Volumes and Applications, *Quart. J. Math. Oxford Ser.* 48 (1997), 235-242.
40. L. Saloff-Coste, A note on Poincaré, Sobolev, and Harnack inequalities, *Internat. Math. Res. Notices* (1992), 27-38.

41. L. Saloff-Coste, Uniformly elliptic operators on Riemannian manifolds, *J. Diff. Geom.* 36 (1992), 417-450.
42. L. Saloff-Coste, Aspects of Sobolev-type inequalities. London Mathematical Society Lecture Note Series, 289. Cambridge University Press, Cambridge, 2002.
43. B.-Y. Song, G.-F. Wei, G.-Q. Wu, Monotonicity formulas for Bakry-Émery Ricci curvature, arXiv:1307.0477.
44. N. Varopoulos, The Poisson kernel on positively curved manifolds, *J. Funct. Anal.* 44 (1981), 359-380.
45. G.-F. Wei, W. Wylie, Comparison geometry for the Bakry-Émery Ricci tensor, *J. Diff. Geom.* 83 (2009), 377-405.
46. J.-Y. Wu, Upper Bounds on the First Eigenvalue for a Diffusion Operator via Bakry-Émery Ricci Curvature II, *Results Math.* 63(2013), 1079-1094.
47. J.-Y. Wu,  $L^p$ -Liouville theorems on complete smooth metric measure spaces, *Bull. Sci. Math.* 138 (2014), 510-539.
48. J.-Y. Wu, P. Wu, Heat kernels on smooth metric measure spaces with nonnegative curvature, arxiv.1401.6155.
49. P. Wu, On the potential function of gradient steady Ricci solitons, *J. Geom. Anal.* 23 (2013), 221-228.
50. S.-H. Zhu, The comparison geometry of Ricci curvature. In: Comparison geometry (Berkeley, CA, 1993-94), volume 30 of Math. Sci. Res. Inst. Publ, pages 221-262. Cambridge Univ. Press, Cambridge, 1997.

DEPARTMENT OF MATHEMATICS, SHANGHAI MARITIME UNIVERSITY, HAIGANG AVENUE 1550, SHANGHAI 201306, P. R. CHINA

DEPARTMENT OF MATHEMATICS, CORNELL UNIVERSITY, ITHACA, NY 14853, UNITED STATES  
*E-mail address:* `jywu81@yahoo.com`

DEPARTMENT OF MATHEMATICS, CORNELL UNIVERSITY, ITHACA, NY 14853, UNITED STATES  
*E-mail address:* `wupenguin@math.cornell.edu`